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THE EQUATIONS OF THE MOVEMENT OF THE SPHERICAL PENDULUM THROUGH THE LAGRANGIAL MECHANICS

AS EQUAÇÕES DO MOVIMENTO DO PÊNDULO ESFÉRICO POR MEIO DA MECÂNICA LAGRANGIANA E HAMILTONIANA

LAS ECUACIONES DEL MOVIMIENTO DEL PÉNDULO ESFÉRICO MEDIANTE LA MECÁNICA LAGRANGIANA Y HAMILTONIANA

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ABSTRACT
Newtonian Mechanics, formulated by Isaac Newton, was, until the mid-eighteenth century, the best tool available for studying dynamical problems. However, this required more coordinates to work than necessary. The study carried out by Lagrange, at the end of the 18th century, aimed to make the method of obtaining the equations of motion of natural phenomena established in the social and scientific sphere more simple and elegant. This was done through d'Alembert's Principle and the introduction of generalized coordinates in Analytical Mechanics. Unlike Newtonian Mechanics, Lagrangian Mechanics eliminates any reference to bonding forces, which brings us great advantage, since in most cases we do not immediately know the expressions that define bonding forces. The objective of this text is to present, in a simple and concise way, the lagrangian mechanics and to show the equations of the spherical pendulum's motion.

Keywords: Lagrangian Mechanics. Hamiltonian Mechanics. Equations of Motion. Spherical Pendulum.

RESUMO
A Mecânica newtoniana, formulada por Isaac Newton, era, até meados do século XVIII, a melhor ferramenta que se tinha para estudar problemas da dinâmica. No entanto, esta exigia mais coordenadas para se trabalhar do que o necessário. O estudo feito por Lagrange, ao final do século XVIII, tinha como objetivo tornar mais simplório e elegante o método de se obter as equações de movimento dos fenômenos naturais estabelecidos do âmbito social e científico. Isso foi feito através do Princípio de D'Alembert e da introdução das coordenadas generalizadas na Mecânica Analítica. Ao contrário da Mecânica newtoniana, a Mecânica lagrangiana elimina qualquer referência às forças de vínculo, o que nos traz grande vantagem, pois na maioria dos casos não sabemos de imediato as expressões que definem as forças de vínculo. O objetivo deste texto é apresentar, de maneira simplória e concisa, a Mecânica lagrangiana e exibir as equações do movimento do pêndulo esférico.

Palavras-chave: Mecânica Lagrangiana. Mecânica Hamiltoniana. Equações do Movimento. Pêndulo Esférico.

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RESUMEN
La mecánica newtoniana, formulada por Isaac Newton, fue, hasta mediados del siglo XVIII, la mejor herramienta disponible para estudiar los problemas de la dinámica. Sin embargo, requería más coordenadas de las necesarias para trabajar. El estudio realizado por Lagrange, a finales del siglo XVIII, tenía como objetivo hacer más simple y elegante el método para obtener las ecuaciones de movimiento de los fenómenos naturales en el ámbito social y científico. Esto se logró a través del Principio de D'Alembert y de la introducción de las coordenadas generalizadas en la Mecánica Analítica. A diferencia de la mecánica newtoniana, la mecánica lagrangiana elimina cualquier referencia a las fuerzas de vínculo, lo que representa una gran ventaja, ya que en la mayoría de los casos no conocemos de inmediato las expresiones que definen dichas fuerzas. El objetivo de este texto es presentar, de manera sencilla y concisa, la mecánica lagrangiana y mostrar las ecuaciones de movimiento del péndulo esférico.

Palabras clave: Mecánica Lagrangiana. Mecánica Hamiltoniana. Ecuaciones del Movimiento. Péndulo Esférico.

1 INTRODUCTION

The Mechanical models in which Newton's Mechanics refined in certain aspects, bringing all the mathematical foundation, and are pillars of Analytical Mechanics, were Lagrangian Mechanics and Hamiltonian Mechanics. These are consistent with Newtonian Mechanics, both constituting more abstract formulations of the latter. The Lagrangian and Hamiltonian formalisms are based on energetic propositions, involving the kinetic energy and the potential energy of Newtonian systems, from which several mechanical variables for such systems can be derived by variational methods. From a theoretical point of view, they allow a deeper understanding of Newton's Mechanics; from the heuristic point of view, they allow, for some systems, simpler and more immediate solutions than those achieved with the usual formalism of Newtonian Mechanics.

We have subtly emphasized the elementary principles of Analytical Mechanics. The pioneering spirit of Lagrangian Mechanics is notoriously given to the Italian mathematician Joseph Louis Lagrange (1736-1813), whose formalism considers a scalar function: called Lagrangian, which characterizes the system that models the event studied.

Thus, a system can be described by Lagrange's equations, which are n (degrees of freedom) differential equations of the second order and invariant under a coordinate transformation. Some of the advantages over the Newtonian formulation is that, in addition to the invariant form of the equations, the forces are literally derived from the Lagrangian function.

Let us see below the elementary principles of this theory.

2 LAGRANGIAN MECHANICS

The core of this section is to expose, in a simplistic way, the formulation of Mechanics developed by the Italian mathematician Joseph Louis Lagrange (1735-1813). Before starting the theory, allow us to inform you that the text presented here is based on the references Júnior (2018), Lemos (2007), Neto (2004), Soldovieri (2013) and Villar (2015).

Definition 1. (Bond). A link is the condition that the coordinates that describe the system can be subject to.

It should be noted that such constraints, i.e., the bonds, must be taken into account even before the equations of the system's motion are formulated.

Definition 2. (Degrees of freedom). We call degrees of freedom the number of independent quantities that determine, in a unique way, the position of a system.

For a mechanical system consisting of m bodies, placed in a space of dimension k and containing p bonds, the number of degrees of freedom is given by:

$$n = km - p \qquad (1)$$

Definition 3. (Generalized coordinates). We call generalized coordinates the minimum set of coordinates capable of completely characterizing the position of a system.

The number of degrees of freedom is equal to the number of generalized coordinates. Therefore, it is possible to introduce generalized coordinates n such that the position vector of the ith particle is written as being and the equations of bonds are identically satisfied.

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, \ldots, q_n, t), \quad i = 1, 2, \ldots, m, \qquad (2)$$

We call the system configuration space the Cartesian space that has generalized coordinates as its coordinate axes.

Definition 4. (Lagrange or Lagrangian function). The Lagrangian or Lagrangian function of a mechanical system is defined by where is the generalized coordinate vector, the generalized velocity vector, T and V are the kinetic and potential energies, respectively.

Once the Lagrange function is defined, we can obtain the equations of motion of a mechanical system. Let's see how to obtain such equations, making use of an important result of Variational Calculus called the Principle of Least Action or Hamilton's Principle.

Definition 5. (Principle of least action or Hamilton's Principle). Given a mechanical system described by the Lagargian, its motion from instant t1 to instant t2 is such that the action is minimal (more generally, stationary) for the actual trajectory, keeping the start and end points of the trajectory fixed in the configuration space.

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t)dt \qquad (3)$$

Our goal now is to answer the following question: What condition must we have for action S to be minimal?

Well, let's consider the variations

$$\bar{q}_i = q_i(t) + \delta q_i(t), \quad \text{com} \quad i = 1, 2, \ldots, n, \qquad (4)$$

Where:

$\delta q' s$ the are independent of each other and arbitrary, satisfying the condition, for all The variation of the action S is given by

Imposing $\delta S = 0$ and doing some algebraic manipulations, we conclude that (1) is equivalent to (2). These latter equations are called the Euler–Lagrange equation.

Note 1. The Euler–Lagrange equations are the equations of motion of the system and constitute a system of n EDO's of the 2nd order.

To determine the dynamics of a system, according to the Lagrangian theory, we must follow the following steps:

1: Choose the generalized coordinates

2: Calculate the kinetic energies and potentials T and V , respectively, to determine the Lagrangian These should be expressed as a function of the only;

3: Calculate the partial derivatives and substitute them in (2).

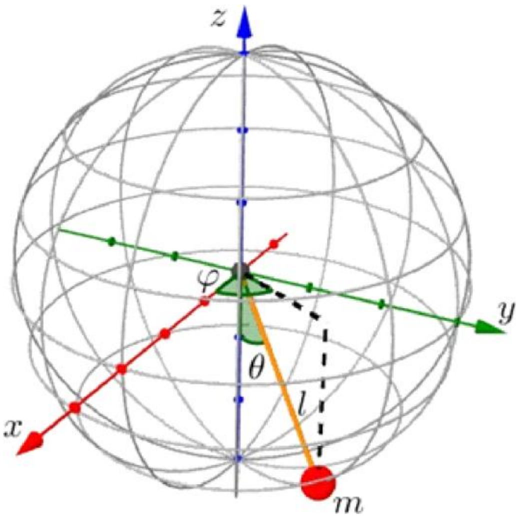
Next, we will use the theory exposed in this section to find the equations of the Movement of the spherical pendulum.

3 THE EQUATIONS OF SPHERICAL PENDULUM MOTION

Consider a rod of negligible mass, of length l and with one end attached to the origin of Euclidean space $\mathbb{R}^3$. Suppose a mass attached m to the free end of the rod, moving in space, as illustrated in the figure below.

Figure 1

Spherical pendulum



Source: Prepared by the author

Note that this problem has 1 link, which is precisely the length l of the pendulum rod. In this way, the number of degrees of freedom and, consequently, of generalized coordinates of the system is

$$n = 3 \cdot 1 - 1 = 2. \qquad (5)$$

We then choose, as generalized coordinates, the angles $q_1 = \varphi$ and $q_2 = \theta$as illustrated in Figure 1. Thus, the position of the mass m is given by:

$$\begin{cases} x = l\sin(\theta)\cos(\varphi) \\ y = l\sin(\theta)\sin(\varphi) \\ z = -l\cos(\theta) \end{cases} \qquad (6)$$

By deriving (3) temporally, we obtain

Let us calculate the kinetic and potential energies in order to obtain the Lagrangian function associated with the problem. Well, the kinetic energy is given by the formula. Potential energy, on the other hand, is given by expression

Therefore, the Lagrangian of the spherical pendulum is written

Since we have two generalized coordinates, we will then have two Lagrange equations. Let us first calculate the Lagrange equation for calculating the respective derivatives, we find: θ

While. In the end

Thus, by (5) and (6), the Lagrange equation in is, θ

Or, equivalently,

$$\ddot{\theta} + \left(\frac{g}{l} - \dot{\varphi}^2 \cos(\theta)\right) \sin(\theta) = 0. \qquad (7)$$

Now, let's find the Lagrange equation for . Note that φ

$$\frac{\partial L}{\partial \varphi} = 0, \qquad (8)$$

Because the Lagrangian does not depend on Calculating the respective derivatives in φ. terms

$$\frac{\partial L}{\partial \dot{\varphi}} = \frac{1}{2} m l^2 2 \dot{\varphi} \sin^2(\theta) = m l^2 \dot{\varphi} \sin^2(\theta). \qquad (9)$$

Deriving (9) in relation to time, we have that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = m l^2 \ddot{\varphi} \sin^2(\theta). \qquad (10)$$

Notice that in (10) we do not derive the sine, because in this case the angle is constant. Therefore, by θ (9) and (10), the Lagrange equation for is:φ

Therefore, the equations of spherical pendulum motion are described, as follows:

$$\begin{aligned} \ddot{\theta} + \left(\frac{g}{l} - \dot{\varphi}^2 \cos(\theta)\right) \sin(\theta) &= 0, \\ m l^2 \ddot{\varphi} \sin^2(\theta) &= 0. \end{aligned} \qquad (11)$$

4 THE HAMILTONIAN FORMULATION OF THE SPHERICAL PENDULUM

Hamiltonian Mechanics, developed by William Rowan Hamilton (1805-1865), offers an alternative and profoundly enriching formulation for the description of mechanical systems. Whereas the Lagrangian formulation describes dynamics in terms of second-order differential equations in configuration space, Hamiltonian formulation transforms the problem into first-order differential equations in phase space, revealing deep mathematical structures and facilitating qualitative analysis of motion.n2n

4.1 FOUNDATIONS OF HAMILTONIAN FORMALISM

The starting point for the construction of Hamiltonian formalism is the Lagrangian function $L(q, \dot{q}, t)$. The generalized moment combined with the coordinate q_i is defined by:

$$p_i = \frac{\partial L}{\partial \dot{q}_i}. \qquad (12)$$

The transition from Lagrangian to Hamiltonian formalism is accomplished through a Legendre transformation, which exchanges variables $\dot{q}_i$ for moments p_i. Hamiltonian is then defined as:

$$H(\mathbf{q}, \mathbf{p}, t) = \sum_{i=1}^{n} p_i \dot{q}_i - L(\mathbf{q}, \dot{\mathbf{q}}, t). \qquad (13)$$

where velocities $\dot{q}_i$ must be expressed in terms of the moments p_i and coordinates q_i using the definitions of generalized moments.

The equations of motion in Hamiltonian formalism, known as Hamilton's canonical equations, are:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad i = 1, \ldots, n \qquad (14)$$

For conservative systems, where generalized forces derive from a time-independent potential, the Hamiltonian coincides with the total energy of the system: .H = T + V

4.2 HAMILTONIAN FORMULATION FOR THE SPHERICAL PENDULUM

Returning to the problem of the spherical pendulum, we have the Lagrangian one established in the previous section:

$$L = \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}ml^2\dot{\varphi}^2\sin^2\theta + mgl\cos\theta. \qquad (15)$$

The generalized moments conjugated to the coordinates θ and φ are:

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} \quad \Rightarrow \quad \dot{\theta} = \frac{p_\theta}{ml^2} \qquad (16)$$

Applying the Legendre transformation, we get the Hamiltonian:

Replacing the expressions for θ and ̇φ ̇ in terms of the moments:

$$H = p_\theta \cdot \frac{p_\theta}{ml^2} + p_\varphi \cdot \frac{p_\varphi}{ml^2\sin^2\theta} - \left[\frac{1}{2}ml^2\left(\frac{p_\theta}{ml^2}\right)^2 + \frac{1}{2}ml^2\left(\frac{p_\varphi}{ml^2\sin^2\theta}\right)^2\sin^2\theta - mgl\cos\theta\right]. \qquad (17)$$

Carefully simplifying each term:

- Term in:

$$p_\theta : \frac{p_\theta^2}{ml^2} - \frac{1}{2}\frac{p_\theta^2}{ml^2} = \frac{1}{2}\frac{p_\theta^2}{ml^2} \qquad (18)$$

- Term in
- Potential term:

$$-(-mgl\cos\theta) = +mgl\cos\theta. \qquad (19)$$

The final Hamiltonian for the spherical pendulum is:

$$H(\theta, \varphi, p_\theta, p_\varphi) = \frac{p_\theta^2}{2ml^2} + \frac{p_\varphi^2}{2ml^2\sin^2\theta} - mgl\cos\theta. \qquad (20)$$

4.3 HAMILTON'S EQUATIONS FOR THE SPHERICAL PENDULUM

Applying Hamilton's canonical equations, we get the complete dynamical system.

For the coordinate θ and its momentum p_θ:

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2}, \tag{21}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -\left[-\frac{p_\varphi^2 \cos\theta}{ml^2 \sin^3\theta} + mgl\sin\theta\right] = \frac{p_\varphi^2 \cos\theta}{ml^2 \sin^3\theta} - mgl\sin\theta. \tag{22}$$

For the coordinate and its momentum p_φ:

$$\dot{\varphi} = \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{ml^2 \sin^2\theta}, \tag{23}$$

$$\dot{p}_\varphi = -\frac{\partial H}{\partial \varphi} = 0. \tag{24}$$

The last equation reveals an important feature of the system p_φ it is a constant of motion, which is associated with the azimuthal symmetry of the problem.

4.4 EQUILIBRIUM POINTS OF THE HAMILTONIAN SYSTEM

The equilibrium points of the system correspond to the points where all temporal derivatives cancel each other out $\dot{\theta} = 0$, $\dot{\varphi} = 0, \dot{p}_\theta = 0, \dot{p}_\varphi = 0$.

From the equation $\dot{\theta} = 0$, we have $p_\theta = 0$.

From the equation $\dot{\varphi} = 0$, with $\sin^2\theta \neq 0$, we have $\sin^2\theta \neq 0$,

From the equation $\dot{p}_\theta = 0$, with $p_\varphi = 0$ we get:

$$-mgl\sin\theta = 0 \quad \Rightarrow \quad \sin\theta = 0. \tag{25}$$

The solutions are $\theta = 0$ e $\theta = \pi$.

The equation $\dot{p}_\varphi = 0$ is identically satisfied.

Therefore, the break-even points are:

- Lower balance: $(\theta, p_\theta, \varphi, p_\varphi) = (0,0,\varphi_0,0)$.

- Upper balance: $(\theta, p_\theta, \varphi, p_\varphi) = (\pi, 0, \varphi_0, 0)$.

Where φ_0 is an arbitrary azimuthal angle, reflecting the rotational symmetry of the system.

4.5 LINEAR STABILITY OF THE EQUILIBRIUM POINTS

Linear stability analysis is performed by studying the behavior of the system in the vicinity of equilibrium points. We linearize Hamilton's equations by calculating the Jacobian matrix of the vector system $\dot{\mathbf{x}} = (\dot{\theta}, \dot{\varphi}, \dot{p}_\theta, \dot{p}_\varphi)^T$.

4.5.1 Lower equilibrium point ($\boldsymbol{\theta = 0}$)

Expanding Taylor's serial Hamiltonian around $\theta = 0$ and considering $p_\varphi = 0$ we get:

$$H \approx \frac{p_\theta^2}{2ml^2} + \frac{1}{2}mgl\theta^2. \qquad (26)$$

The linearized system in the variables (θ, p_θ) is:

$$\begin{cases} \dot{\theta} = \dfrac{p_\theta}{ml^2} \\ \dot{p}_\theta = -mgl\theta. \end{cases} \qquad (27)$$

In matrix form.

The eigenvalues of this matrix satisfy:

$$\det\begin{pmatrix} -\lambda & \frac{1}{ml^2} \\ -mgl & -\lambda \end{pmatrix} = \lambda^2 + \frac{g}{l} = 0. \qquad (28)$$

$$\lambda = \pm i\sqrt{\frac{g}{l}}. \qquad (29)$$

Purely imaginary eigenvalues indicate that the equilibrium point is a center in the linear approximation, characterizing linear stability.

4.5.2 Upper breakeven point ($\theta = 0$)

Introducing the variation $\theta = \pi + \eta$, with small, and expanding the Hamiltonian:θ=π+ηη

$$\cos\theta \approx -1 + \frac{\eta^2}{2}, \quad \sin\theta \approx -\eta. \qquad (30)$$

For $p_\varphi = 0$, the linearized Hamiltonian is:

$$H \approx \frac{p_\theta^2}{2ml^2} - \frac{1}{2}mgl\eta^2. \qquad (31)$$

The linearized system in $p_\varphi = 0,$ is:

$$\begin{cases} \dot{\eta} = \dfrac{p_\theta}{ml^2} \\ \dot{p}_\theta = mgl\eta. \end{cases} \qquad (32)$$

In matrix form:

$$\frac{d}{dt}\begin{pmatrix} \eta \\ p_\theta \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{ml^2} \\ mgl & 0 \end{pmatrix}\begin{pmatrix} \eta \\ p_\theta \end{pmatrix}. \qquad (33)$$

The eigenvalues satisfy:

$$\det\begin{pmatrix} -\lambda & \frac{1}{ml^2} \\ mgl & -\lambda \end{pmatrix} = \lambda^2 - \frac{g}{l} = 0. \qquad (34)$$

$$\lambda = \pm\sqrt{\frac{g}{l}}. \qquad (35)$$

Real eigenvalues, one positive and one negative, characterize a saddle point, indicating linear instability.

4.6 PHYSICAL INTERPRETATION

Hamiltonian analysis reveals deep structures of the spherical pendulum:

1. The Hamiltonian H represents the total energy of the system and is conserved during movement.
2. The coordinate φ is cyclic, resulting in the conservation of azimuthal angular momentum p_φ.
3. The four-dimensional phase space $(\theta, \varphi, p_\theta, p_\varphi)$ provides a complete description of the dynamics of the system.
4. The existence of two equilibrium points with distinct natures - a stable center (lower point) and an unstable saddle point (upper point) - organizes the overall structure of phase space.
5. For $(\theta, \varphi, p_\theta, p_\varphi)$, new equilibrium points corresponding to precessional movements emerge, further enriching the dynamics of the system.

The Hamiltonian formulation not only provides an elegant and symmetrical description of dynamics, but also paves the way for advanced methods of analysis, such as perturbation theory and the study of Hamiltonian chaos, which can be explored in future investigations into more complex dynamical systems.

5 FINAL CONSIDERATIONS

In general, at the end of this research, we acquired comprehensive knowledge in important topics from Analytical Mechanics, Analysis with multiple variables, qualitative EDO, having as motivation the mathematical formulation of the equations of the movement of the spherical pendulum.

Therefore, in order to be successful in the research, studies were developed directed to Analytical Mechanics, more specifically Lagrangian Mechanics. In these studies, we can understand Lagrange's objective in developing this rich theory, which was to make the study of the dynamics of a mechanical system simpler than what was proposed by Isaac Newton.

With the necessary theory, we were able to formulate the spherical pendulum problem and obtain its equations of motion, where we found two equations. These depend explicitly on the angles that determine the positions of the mass m. We aim to later make a study about the stability of the system studied.

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