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NONLINEAR PENDULUM: APPLICATIONS OF DIFFERENTIAL EQUATIONS IN COMPUTER SIMULATION AND SOCIO-ENVIRONMENTAL REFLECTIONS

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ABSTRACT
This study uses the mathematical modeling of a pendulum to teach differential equations, combining computer simulation and socio-environmental reflection. Representing a dynamic system that acquires complexity as oscillations increase, the pendulum allows students to explore different behaviors from conservative (linear regime) to chaotic. The qualitative methodology, with participant observation and interviews, uses tools such as the Fast Fourier transform for the purpose of analyzing the nonlinear phenomenon and the Runge-Kutta method for simulating several scenarios (linear/nonlinear), allowing to project the use of the methodology for the purpose of practical and critical analysis of phenomena such as climate change and use of natural resources. Inspired by Critical Mathematics Education (CME), the study seeks to form critical citizens who see mathematics as a tool to interpret and transform reality, in line with the idea of "mathematics in action" defended by Freire and Skovsmose. Thus, it is concluded that the nonlinear pendulum modeling contributes both to the teaching of differential equations and to the development of socio-environmental awareness and education focused on social justice.

Keywords: Mathematical Modeling, Critical Mathematics Education, Chaotic Behavior, Educational Tool, Bioinputs.

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INTRODUCTION

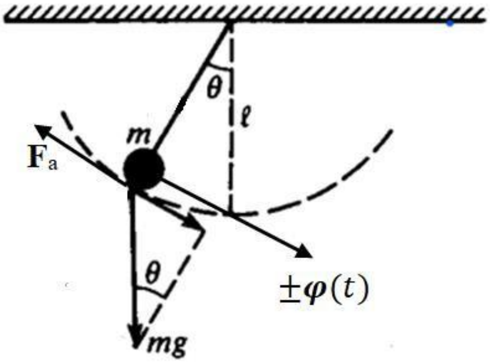
The study of dynamical systems and their characteristics, both linear and nonlinear, has proven to be fundamental in several areas of science and technology, such as Physics, Engineering and Mathematics itself. One of the most emblematic systems in this context is the simple pendulum, widely used to illustrate oscillatory phenomena. When small oscillations are considered, the system can be described as linear, with predictable periodic behavior. However, when major oscillations or external excitations are introduced, nonlinear behavior emerges, resulting in complex, often chaotic dynamics (Savi, 2006).

The pendulum, which consists of a mass attached to an inextensible wire coupled to a fixed point, is governed by the nonlinear differential equation (1), which describes its motion:

$$ml\frac{d^2\theta}{dt^2} + cl\frac{d\theta}{dt} + mgsen(\theta) = \varphi(t) \qquad \text{(equação 1)}$$

where m is the mass of the sphere [kg], l the length of the wire [m], c the damping constant [kg/s], g the gravitational acceleration [m/s²], θ the angular displacement [rad] and φ(t) the external excitation force, as explained in figure 1.

Figure 1: Simple nonlinear pendulum.



Source: Adapted from Leithold (1982).

In this work, air resistance is considered, which introduces a damping factor to the system. The force of resistance is proportional to the angular velocity of the pendulum and described by equation (2):

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$$\boldsymbol{F}_a = -c\frac{d\theta}{dt} \qquad \text{(equação 2)}$$

From these equations, the behavior of the system can be studied in different scenarios. When it comes to small oscillations, the equation can be linearized and the motion is approximated by a sinusoidal function. However, by abandoning the hypothesis of small displacements, nonlinearities arise, leading to more complex and often unpredictable behaviors, as described by the numerical simulation of the Runge-Kutta method.

Mathematical modeling via computer simulation, such as the one used in this study, not only allows the detailed analysis of dynamic systems, but also expands the possibilities of exploring phenomena that, in the physical world, could be difficult to reproduce. In addition, the use of tools such as the Fast Fourier Transform (FFT) (Spilsbury, 2016) allows capturing the evolution of nonlinear phenomena over time, showing how small perturbations can induce large variations in the system, as shown in figure 1.

The relevance of these models goes beyond technical applications. In a Critical Mathematics Education (CME) context, dynamical systems modeling can be used to discuss social, environmental, and sustainability issues (Skovsmose, 2001). For example, the swing of a pendulum can be analogously compared to the swing of natural resources or even to the spread of disease, as in epidemics. The computer simulation of nonlinear scenarios, such as the study of the pendulum with external excitation, allows students to reflect on the complex interactions present in socio-environmental phenomena, that is, how changes in physical, geometric, initial/boundary conditions and, especially, external forces can significantly affect the behavior of the dynamic systems that model the most diversified systems, among them the socio-environmental ones.

Thus, this work proposes to investigate how the mathematical modeling of the nonlinear pendulum, using differential equations and computer simulation, can be applied in the teaching of differential equations to promote a critical reflection on socio-environmental issues. The nonlinear pendulum is a valuable pedagogical tool for understanding differential equations and mechanical systems, and also for stimulating students' critical awareness of how mathematics can be used to analyze and solve real and complex problems.

From CME's perspective, teaching differential equations through models such as the nonlinear pendulum offers a unique opportunity to interconnect mathematics with ethical, social, and environmental issues. The incorporation of themes such as sustainability and

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social justice, as advocated by theorists such as Paulo Freire, contributes to a mathematics education that goes beyond the technical domain, promoting the development of a critical and reflective consciousness, necessary for social transformation.

NONLINEAR PENDULUM AND DIFFERENTIAL EQUATIONS: OPPORTUNITIES FOR REFLECTION

Mathematical modeling can be a valuable tool in teaching differential equations because it enables students to understand complex phenomena through mathematical representations. The nonlinear pendulum, in particular, stands out as a relevant system to enable such an approach due to its complex dynamics and applicability in real contexts.

Recent studies have explored the complexities associated with nonlinear pendulum behavior, offering opportunities to foster critical reflection on socio-environmental issues. However, there is a gap in the pedagogical application of these concepts in the teaching of differential equations (Jia et al., 2020; Rahimi Dolatabad et al., 2022; Kundu; Chatterjee, 2022; Labetoulle; Savadkoohi; Gourdon, 2022; Arango, 2021; Ochkov et al., 2023). While these studies provide valuable reflections on nonlinear dynamics and utilize advanced theoretical and numerical approaches, they do not delve into pedagogical strategies to integrate these findings into the educational context, especially to promote critical reflections on complex dynamic systems and socio-environmental issues.

To fill this gap, several methodologies have been proposed, emphasizing approaches that promote critical thinking and meaningful learning. "Contextual Learning" stands out for connecting mathematical concepts to students' life experiences, allowing them to relate mathematical problems to real situations (Khotimah; Masduki, 2016). This approach promotes a deeper and more critical understanding of knowledge, encouraging students to reflect on the socio-environmental impact of their actions.

The use of digital technologies and computer simulations has been shown to be a way to enrich the teaching of differential equations and mathematical modeling. These tools provide interactive and adaptive learning environments, facilitating students' active participation and fostering deeper understanding through experimentation and model refinement. The integration of simulations at different levels of automation helps students transition from manual to digital activities, strengthening algorithmic and modeling concepts (Grandgenett et al., 2000; Greubel et al., 2022; Cevikbas; Greefrath; Siller, 2023). In addition, the use of advanced technologies, such as physics-informed neural networks and surrogate modeling based on active learning, expands the possibilities of simulating complex nonlinear systems in the educational context (Moya; Lin, 2021; Kapadia; Feng; Benner, 2023).

Integrating technological resources, such as simulation and graphic visualization software, is another effective strategy. The use of GeoGebra, for example, facilitates the exploration of multiple representations and balances analytical, graphical, and numerical approaches in the teaching of differential equations (Igliori; Almeida, 2017). This technological integration allows students to interact with dynamic models, such as the nonlinear pendulum, promoting learning autonomy and deepening the understanding of the phenomena studied.

The connection between mathematical modeling and understanding of real phenomena can be a way to engage students and develop critical skills. The application of mathematical models to everyday problems, such as pandemics (Meyer; Lima, 2023), evacuations (Greubel et al., 2022) and complex physical systems, allows students to analyze the impact of different variables and policies, promoting critical reflection on public decisions and socio-environmental issues. The modeling of systems such as the nonlinear pendulum can be used to explore natural phenomena and discuss environmental implications, encouraging awareness about sustainability and the use of natural resources (Kapadia; Feng; Benner, 2023; Balseca et al., 2023).

"Problem-Based Learning (PBL)" is another methodology that places students as active agents in their learning process, using real problems as a starting point (Santos et al., 2020). This approach promotes autonomy and reflective thinking, encouraging students to apply their knowledge in practical situations, which is particularly relevant in the study of systems such as the nonlinear pendulum.

The use of mathematical modeling activities in the STEM context[5] has shown a significant increase in students' abilities, encouraging complex problem-solving and collaboration (Armutcu; Bal, 2023; Greubel et al., 2022). In addition, teacher training is highlighted as a crucial element for the effective implementation of these approaches, requiring strategies that consider the specificities of students and their difficulties with the topic (Andresen, 2023; Balseca et al., 2023; Vitória et al., 2021).

[5] The context of STEM education refers to an integrated educational approach that combines the areas of Science, Technology, Engineering, and Mathematics. The primary goal of STEM education is to prepare students to face real-world challenges by developing critical skills for the twenty-first century, such as critical thinking, problem-solving, innovation, and collaboration.

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The application of "non-routine problems" and teacher training to improve students' understanding of differential equations courses are also considered relevant approaches (Bibi et al., 2019). After all, the emphasis on problems that require critical thinking and creative solutions develops students' ability to think critically when facing unprecedented situations, aligning with the principles of critical mathematics education.

The development of teaching materials based on "Higher Order Thinking Skills (HOTS)" aims to promote critical thinking and problem-solving skills (Arfinanti, 2020). This approach challenges students to analyze, evaluate, and create solutions to complex problems, contributing to the formation of intellectual autonomy and a deeper understanding of mathematical concepts.

Mathematical modeling presents itself as a powerful tool for the development of students' critical and reflective thinking. By working with models that involve real phenomena and contextual variables, students are encouraged to critically analyze the impact of different factors and decisions, promoting not only the learning of mathematical concepts, but also awareness of socio-environmental issues (Meyer; Lima, 2023; Armutcu; Bal, 2023; Cevikbas; Greefrath; Siller, 2023).

An "investigative mathematics education" can also contribute to the promotion of learning environments that challenge students to explore fundamental concepts of differential equations, encouraging the development of a critical and questioning view of mathematical concepts (Rogovchenko; Rogovchenko, 2022). In the context of the nonlinear pendulum, investigative tasks may involve the analysis of complex nonlinear behaviors, encouraging students to deepen their understanding and reflect on socio-environmental implications.

While digital technologies offer several advantages, such as improvements in data visualization and solution validation, there are challenges to be addressed. The lack of technological competence among students and teachers and the threat of the "black box", in which students may blindly trust the solutions generated by the tools, can limit effective learning (Cevikbas; Greefrath; Siller, 2023; Andresen, 2023). Therefore, it is urgent to develop pedagogical strategies that promote an understanding of the underlying mathematical principles, avoiding excessive dependence on automatic solutions and promoting students' autonomy in the modeling process (Andresen, 2023; Greubel et al., 2022).

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Innovation in teaching differential equations through the integration of mathematical experiments and digital resources is also emphasized (Zhao, 2022). The emphasis on guided inquiry and independent exploration of problems can develop students' autonomy and capacity for discovery, preparing them to manipulate knowledge in a meaningful and contextualized way.

The use of advanced computational methods, such as physics-informed neural networks and surrogate models based on active learning, has expanded the possibilities of simulation of complex nonlinear systems in the educational context. These methods allow for more efficient and accurate simulations, enabling students to explore complex dynamical systems and understand sensitivity to parametric variations (Moya; Lin, 2021; Kapadia; Feng; Benner, 2023). The integration of these techniques in the teaching of differential equations offers opportunities to deepen the understanding of the modeling of nonlinear systems and to discuss practical applications in areas such as energy and environment (Kapadia; Feng; Benner, 2023; Moya; Lin, 2021).

Using reduced-order models and active learning allows students to better understand how dynamical systems can be accurately modeled and how the use of advanced technology can optimize simulations of complex nonlinear systems (Kapadia; Feng; Benner, 2023). In the case of the nonlinear pendulum, these techniques can be applied to investigate how different variables, such as friction and external forces, affect their behavior, providing an opportunity for students to critically reflect on environmental and energy issues.

As described by the authors cited, there is strong evidence in their studies of the importance of integrating digital technologies, innovative pedagogical methodologies and mathematical modeling in the teaching of differential equations to promote critical reflection on socio-environmental issues. The nonlinear pendulum emerges as an ideal system to make this approach viable, allowing students to explore complex phenomena and develop essential skills to interpret and transform the reality in which they live.

METHODOLOGY

The mathematical modeling of the nonlinear pendulum through computer simulations offers an opportunity for the teaching of differential equations, while promoting a critical reflection on socio-environmental issues. This integrative approach can enrich the learning process by connecting theoretical content with concrete applications of global relevance, providing students with a broader and more contextualized understanding.

In order to investigate how the mathematical modeling of the nonlinear pendulum, using differential equations and computer simulation, can be applied in the teaching of differential equations to promote a critical reflection on socio-environmental issues, a qualitative approach was adopted, as suggested by Bogdan and Biklen (1994). This study was characterized as exploratory qualitative, with the objective of understanding how students appropriate the modeling and simulation of the nonlinear pendulum in the learning process, connecting this knowledge to sustainability issues and other socio-environmental implications.

The study was carried out with an undergraduate class of the Mathematics course of a higher education institution, composed of seven students enrolled in the discipline of Applied Differential Equations I. The sampling was intentional, selecting participants who expressed interest in joining the research and were willing to critically reflect on the application of differential equations in real problems. The group of students was subdivided into 3 disjoint groups of students, one of which was composed of 3 students. Data collection was carried out from multiple sources, as described by Bogdan and Biklen (1994), ensuring a more comprehensive view of the phenomenon investigated. Participant observation was one of the main strategies adopted, in which the researcher, who was also the teacher of the course, followed the students' interactions during classes and recorded their perceptions in a field diary.

Semi-structured interviews were conducted with the seven students, deepening their perceptions about the learning of mathematical modeling and the connection with socio-environmental issues. In addition, documents such as reports and projects developed by the students were analyzed, which evidenced the practical application of the theoretical concepts.

The pedagogical intervention consisted of activities structured in four stages: 1) Theoretical introduction about the nonlinear pendulum and the differential equations involved, focusing on the modeling of oscillatory behavior; 2) Mathematical modeling, in which students developed their own models, using differential equations to describe the behavior of the pendulum; 3) Computer simulation, in which students applied tools such as GeoGebra and MATLAB to simulate different scenarios and analyze the behavior of the pendulum under nonlinear conditions; and 4) Critical reflection, in which the results of the simulations were discussed in a group, relating them to socio-environmental themes, such as the sustainable use of resources and the analysis of complex systems in nature.

All procedures were carried out in accordance with current ethical standards, and the Informed Consent Form was signed by all participants. It ensured that students were free to withdraw from the survey at any time, and all information was treated confidentially.

To ensure the validity and reliability of the results, data triangulation strategies were used, comparing the different collection methods (observations, interviews, and document analysis). The dense description of the observed contexts and the researcher's reflective process about possible biases were also adopted to increase the reliability of the findings.

Among the limitations of the research, the fact that the researcher's performance as a teacher may have influenced the dynamics of the classroom stands out. In addition, the sample size and qualitative nature of the study limit the generalizability of the results.

DATA ANALYSIS AND DISCUSSION

Data analysis followed the content analysis method, as described by Bogdan and Biklen (1994). Initially, an open coding was carried out, in which the data were read and categorized into emerging themes, such as the understanding of the concepts of differential equations, the ability to connect these concepts to real problems, and the development of critical thinking about socio-environmental issues. Then, the categories were refined and organized into central themes, which included: (i) the impact of mathematical modeling on the understanding of dynamic systems, (ii) the use of computer simulations as a pedagogical tool, and (iii) students' critical reflection on environmental and sustainability issues. Finally, the themes were analyzed in the light of the literature, seeking to understand the educational implications of the adopted methodology.

i) The impact of mathematical modeling on the understanding of dynamical systems

In this perspective, the groups of students performed numerical simulations of the system, with different parameters. Each group analyzed the system responses to the different values of external excitation amplitude A, with the freedom to adjust the parameters of mass, damping, wire length and initial conditions. This approach allowed each group to investigate how different combinations of parameters influence the behavior of the system and the transition between linear and nonlinear regimes.

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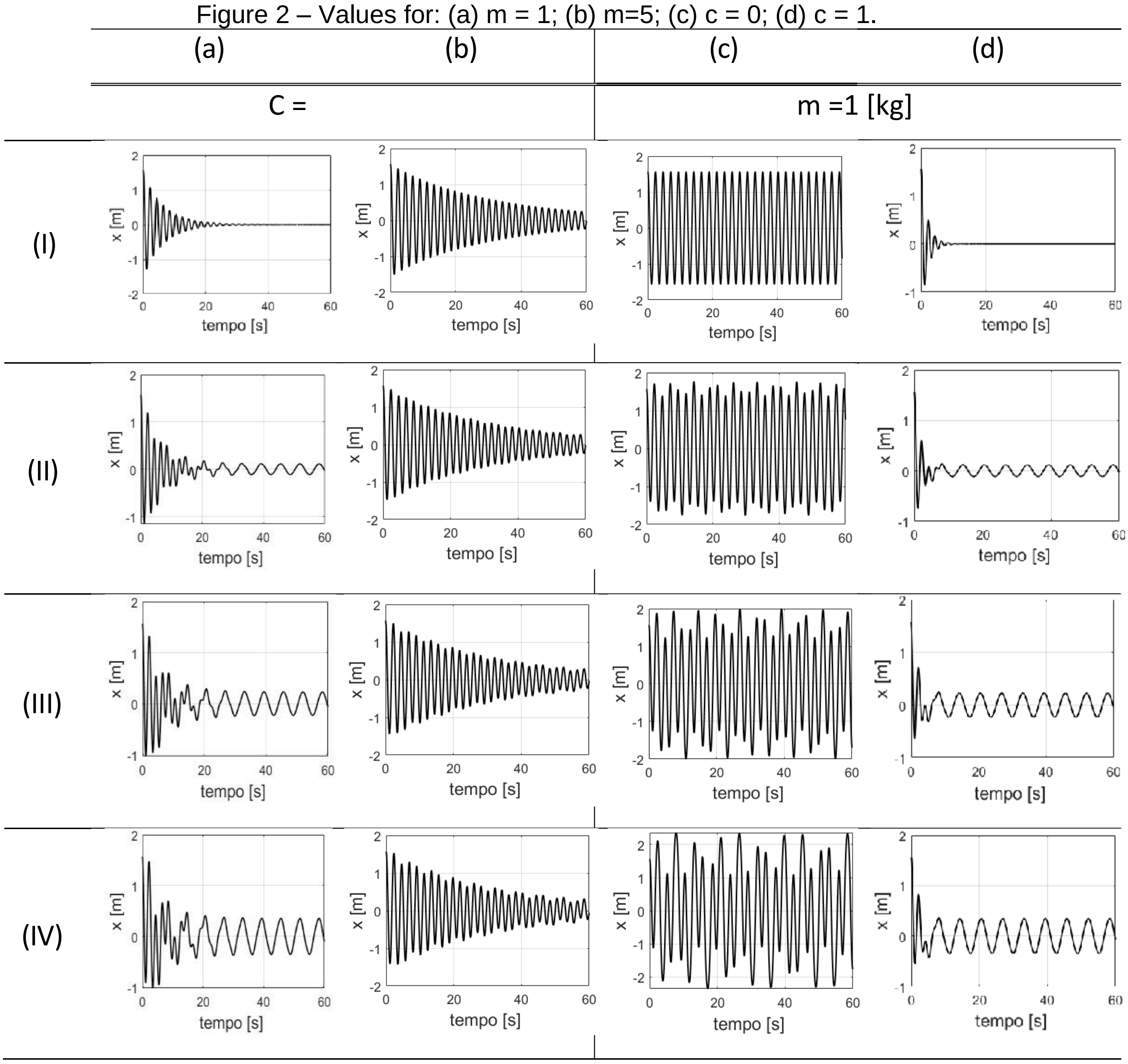
Each group conducted simulations using the indicated excitation amplitude values (A = 0(I), 1(II), 2(III) and 3(IV) N), but freely adjusting the parameters of mass, damping, wire length, and boundary conditions of the system:

* **Group 1:** The students in this group considered changes in the structural damping parameters c [kg/s] and the mass m [kg] of the pendulum, analyzing how these factors affect energy dissipation and oscillatory behavior in different excitation regimes. In this case, the wire length and initial conditions were considered fixed at 1 [m] and meter and the conditions $\theta(0) = \pi/2$, $\dot{\theta}(0) = 1$, respectively, as shown in figure 2.

Figure 2 – Values for: (a) m = 1; (b) m=5; (c) c = 0; (d) c = 1.



Source: Survey data.

In this simulation scenario, the students of the group realized that in the case of the conservative system (I) the system has a predominantly linear behavior independent of the parameters of mass and damping. Considering that non-zero excitation amplitudes the

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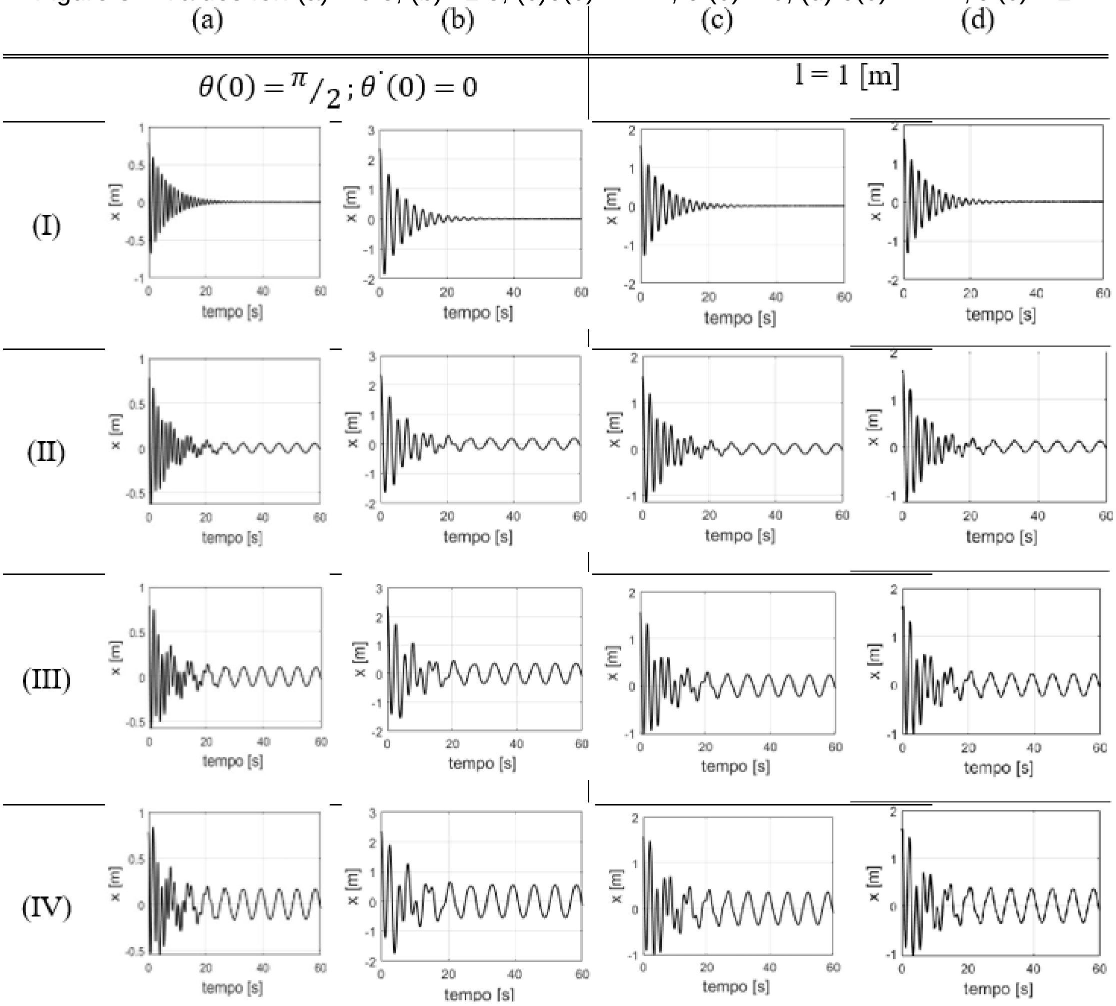
system has a nonlinear behavior inversely to the growth of the mass factor, this can be observed from the comparison of the responses of the system in Figure 2 (II-IVa) and (II-IVb) and, regardless of the damping factor, see Figure 2 (II-IVa,b).

* **Group 2:** The considerations of this group focused on the variation of the wire length and the initial conditions of position and angular velocity, investigating the sensitivity of the system to small changes in the starting conditions. In this case, constant damping and mass equal to ac = 0.3 [kg/s] and mass m = 1 [kg] were considered, respectively, as shown in Figure 3.

Figure 3 – Values for: (a) l=0.5; (b)l=1.5; (c)$\theta(0) = \pi/2$, $\dot{\theta}(0) = 0$; (d) $\theta(0) = \pi/2$, $\dot{\theta}(0) = 1$

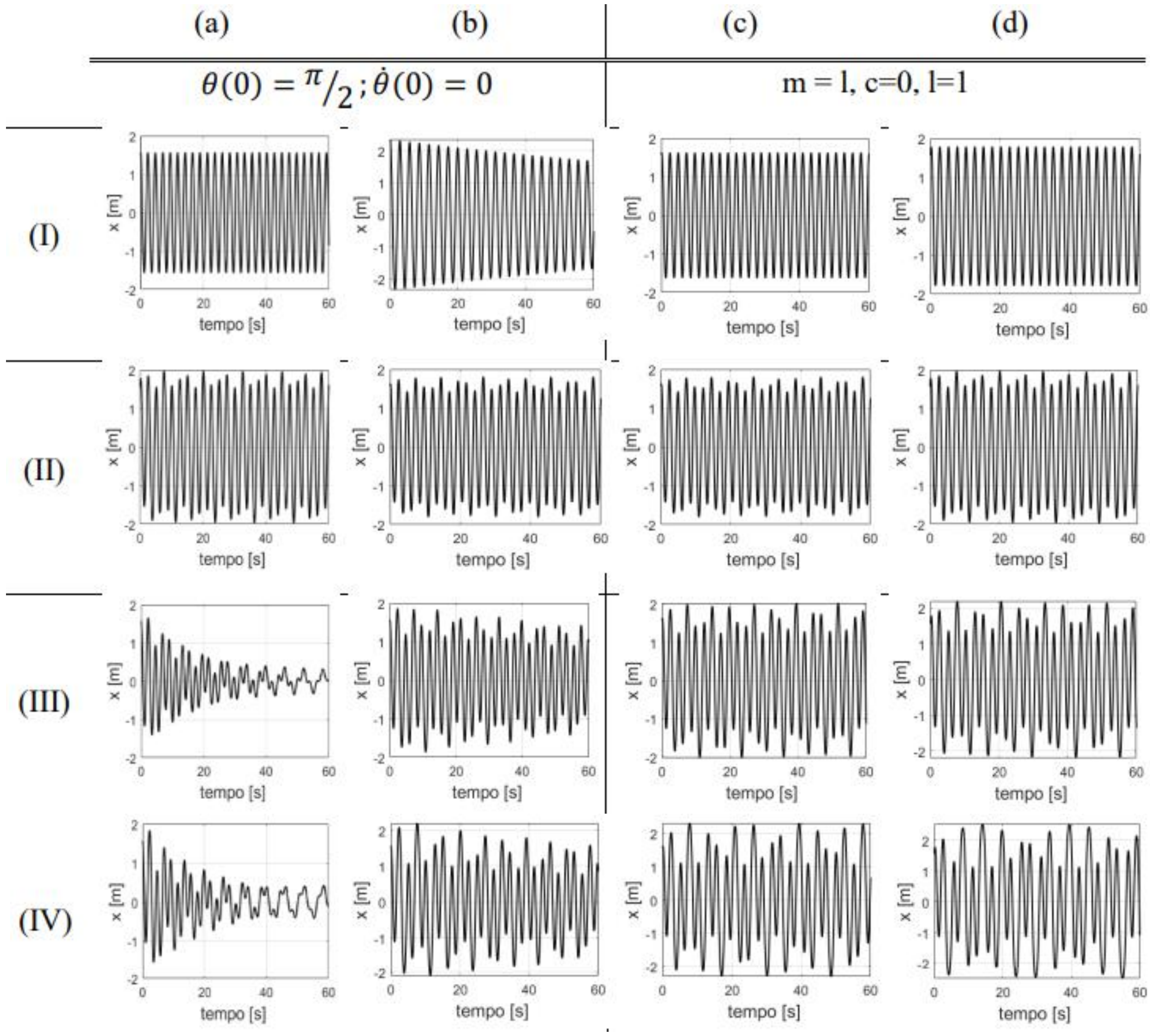


Source: Survey data.

Again, here the students observed that the scenario in which the external excitation is null, basically the change in the length of the wire simply leads to a change in the displacement of the mass Figure 3 (Ia,b) and if no change is observed when the initial conditions are changed, this is consistent with the theory, because in this case the analyzed scenario is purely linear. In the scenarios where the external excitation is non-zero Figure 3 (II-IV) the wire length parameters directly influence the nonlinear effect, however the same is not observed for the boundary conditions, this can be observed if we compare each excitation force amplitude scenario separately for both the wire length parameter (Figure 3 a, b) for boundary conditions, Figure 3 (c,d).

* **Group 3: Finally,** this group explored a broader combination of parameters to observe how adjustments in mass, yarn length and damping, combined with different initial conditions, influence the oscillation pattern, as shown in figure 4.

Figure 4 – Values for: (a) m = l, c=0, l=1; (b) m = l, c=0.01, l=1.5; (c) $\theta(0) = \pi/2$, $\dot{\theta}(0) = 1$;(d) $\theta(0) = \pi/2$; $\dot{\theta}(0) = 2$



Source: Survey data.

The students realized that from the null external excitation condition there is no evidence of nonlinear scenarios regardless of the variation in mass parameters, wire length, damping and initial conditions. However, the direct influence of the parameters on the nonlinear predominance regime is noticeable when an external excitation force is considered, even if small.

In this sense, the results of the numerical simulations allowed each group to observe and identify linear and nonlinear characteristics of the pendulum system. In general, students reported that:

1. For A = 0 [N]: Most of the scenarios showed linear behavior with regular and predictable oscillations. Adjusting parameters such as damping made it easier to see a proportional and less complex response, regardless of the initial conditions or the length of the yarn.
2. For A = 1 and A = 2 [N]: The system showed a transition to nonlinear behavior, with oscillations that deviated from the typical proportionality of linear systems. Variations in the parameters resulted in more dynamic and sensitive responses. The students noted that the inclusion of greater damping, for example, was not enough to offset the effects of external excitation, leading to more complex oscillation patterns.
3. For A = 3 [N]: Each group noted strongly nonlinear features and chaotic behavior, where small changes in parameters or initial conditions caused large variations in responses. This scenario required more careful analysis to identify patterns and understand the complexity of the system.

In addition, numerical simulations provided a valuable tool for the practical understanding of the behavior of the system, leading the students to the conclusion that external excitation offers great influence on the nonlinear effect on the system. However, students faced challenges, especially in relation to:

- Parameter Interpretation: Many students reported difficulties in accurately interpreting the impacts of each parameter on the transition between the linear and nonlinear regimes, mainly due to the number of variables involved.
- Behavior Prediction: The groups that varied widely in the initial conditions and the wire length found it difficult to predict the behaviors of the system under high excitation amplitude (A = 3N), which evidenced the sensitivity of the simulations to small adjustments.

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- Advantages and Disadvantages of Simulations: While simulations have made it easier to visualize the effects of different parameters and initial conditions, they have also had limitations. The absence of a strong theoretical basis made it difficult for some students to interpret the results, and some complex scenarios required extensive and in-depth analysis.

In view of the reports promoted by groups, considering their specificities of analysis parameters and also taking into account the crossings of the analysis of the different groups, according to the items exposed in the items of Parameter Interpretation, Behavior Prediction and Advantages and Disadvantages of the Simulations.

It is observed, especially, the advantages of the methodology in relation to the diversity of situations imposed by the mechanical system studied. This allowed students to discuss numerous additional considerations, allowing a critical analysis of the phenomenon and, consequently, of the mathematical model used. In this way, enabling a critical view of the equation of the mechanical system model, meeting the guidelines of critical mathematics learning, according to Skovsmose (2001).

In order to provide students with support for general conclusions and the closure of partial interpretations of each group on the content of differential equations, all students of the course were asked to perform a simulation with specific parameters, the results of which are illustrated in Figure 5.

Thus, Figure 5 presents the response of the pendulum system illustrated in Figure 1 under different amplitudes of external excitation (A = 0 [N], A = 1 [N], A = 2 [N], A = 3 [N]) in order to explore scenarios of the linear and nonlinear behavior of the system over time. This analysis offers a rich context for the teaching of nonlinear differential equations and the study of complex dynamical systems, in addition to opening space for critical reflection.

In this context, Figure 5 shows the simulated results from the equation of motion (1), considering an object with mass m=1 [kg], damping coefficient c=0 [kg⁄s] and wire length l=1 [m], considering different amplitudes for the external excitation force $\varphi(t) = Asen(wt)$ with the boundary conditions $\theta(0) = \pi$ and $\dot{\theta}(0) = 0$.

ii) The use of computer simulations as a pedagogical tool

Figure 5 (a), (b), (c) and (d) illustrate, respectively, the responses in terms of displacement and velocity as a function of time; the relationship between displacement-

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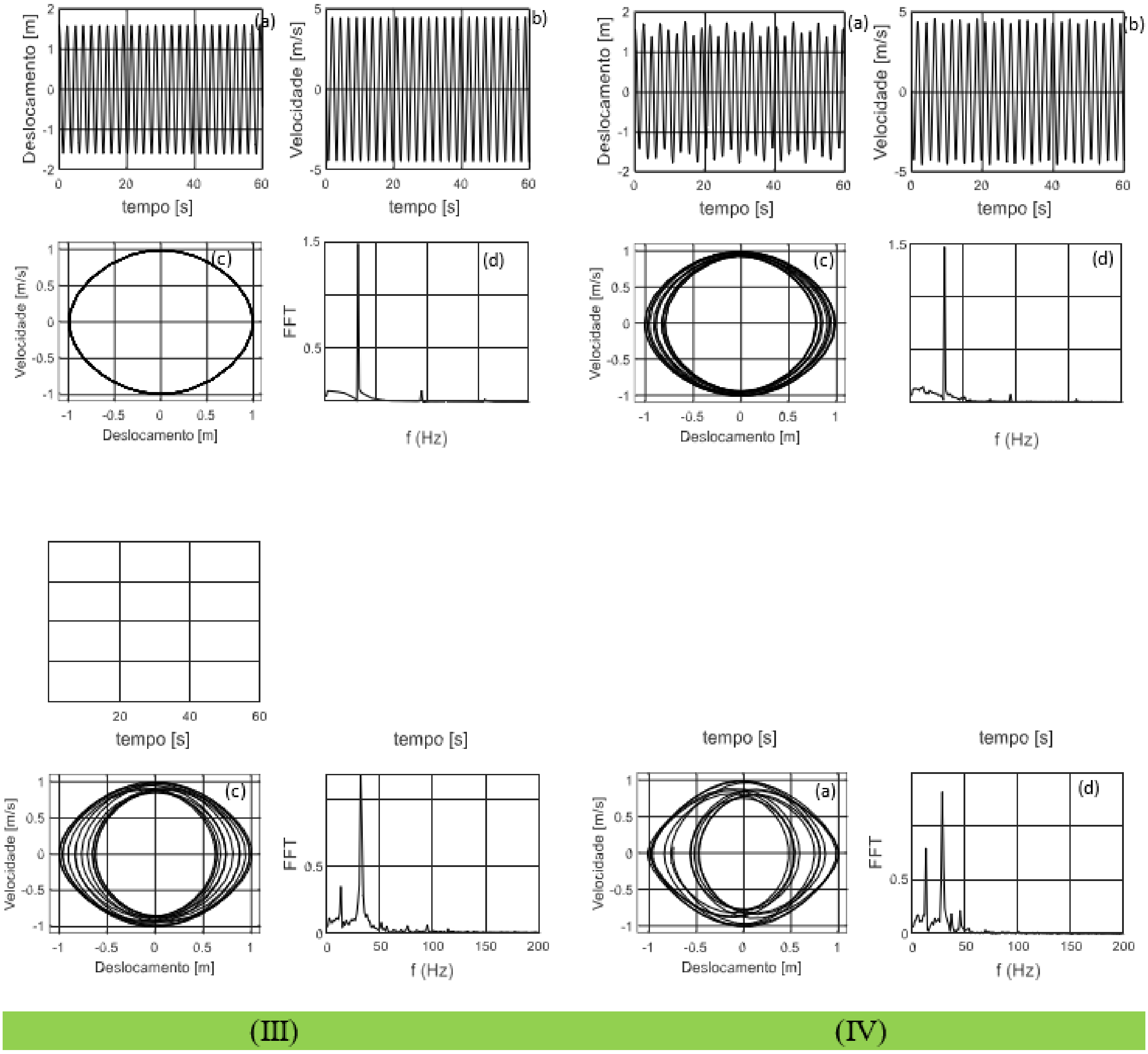
velocity and Fourier transform (FFT) for the four levels of excitation mentioned. These graphs are essential in the study of nonlinear systems, since they show how the variation of the amplitude of external excitation modifies the behavior of the system over time.

Figure 5: Amplitude de excitação externa (I) A = 0 [N]; (II) A = 1 [N]; (III) A = 2 [N]; (IV) A = 3 [N].



Source: Survey data.

These parametric considerations are necessary for the system to provide different behavior scenarios, from linear to nonlinear. In this sense, we seek to illustrate the transition from the pure linear scenario to the nonlinear scenario. The answers obtained in the simulations considered the structural damping condition null.

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Figure 5 presents the results for different external excitation amplitude values, including: (a) Displacement; (b) Speeds; (c) Cause and effect diagram; and (d) FFT of the displacement. Based on items I a-d, the validation of the model defined by the equation of motion (1) and implemented by means of a numerical method is observed. In items I-IV d, it is possible to notice the evolution of nonlinear behavior evidenced by the disturbance in the FFT graph.

Given this, it is possible to analyze the differences between the graphs to facilitate the understanding of the concepts. In this sense, it is worth mentioning:

- ✓ **Graph (a):** Shows zero external excitation (A = 0 [N]). In this scenario, the system is at rest or presents small natural oscillations, showing that, without an external force applied, the system remains close to a state of static equilibrium. This graph is useful for introducing the concept of linear behavior in simple dynamical systems;
- ✓ **Graph (b):** With an excitation of A = 1 [N], the beginning of more accentuated oscillations is noticed, but still in a controlled regime. Here, the nonlinearity of the system begins to manifest itself, with more complex responses and larger displacements compared to graph (a). This is an interesting point to discuss how nonlinear systems can react disproportionately to small increases in external force;
- ✓ **Graph (c):** With A = 2 [N], the behavior of the system becomes even more dynamic, with greater amplitudes of displacement and variations in velocity. In this graph, it is possible to see the transition to a more chaotic behavior, with greater dependence on the initial conditions. This graph serves to illustrate phenomena of resonance and instability, critical aspects in the study of nonlinear differential equations;
- ✓ **Graph (d):** Under a maximum excitation of A = 3 [N], the system exhibits a totally nonlinear behavior, with large amplitude oscillations and significant resonance signals. Here, students can observe how nonlinearity can lead to unpredictable responses and bifurcations, phenomena typically studied in complex dynamical systems.

The comparison between graphs (a), (b), (c) and (d) allowed students to understand how nonlinear differential equations model systems whose responses vary in a way that is not proportional to external forces. This also reinforces the importance of simulating and visualizing the behavior of complex dynamic systems.

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Thus, the use of advanced computational methods, such as physics-informed neural networks and surrogate models based on active learning, expands the possibilities of simulating complex nonlinear systems in the educational context (Kapadia; Feng; Benner, 2023; Moya; Lin, 2021). These techniques allow students to explore the behavior of the nonlinear pendulum in more depth, understanding the sensitivity to parametric variations and reflecting on the socio-environmental implications of these systems. Mathematics in action, in this case, reveals itself in the application of advanced techniques to solve complex problems that have a direct impact on society.

iii) Students' critical reflection on environmental and sustainability issues

From the perspective of Freire's (1996) and Skovsmose's (2007) theories, the mathematical modeling of the nonlinear pendulum can be used as a pedagogical tool to promote critical reflections on socio-environmental issues in the teaching of differential equations. Skovsmose (2007) introduces the concept of "mathematics in action", highlighting that mathematics is not only a representation of reality, but also an agent that shapes and influences social practices and political decisions.

The analysis of nonlinear systems, such as the nonlinear pendulum, allows us to explore complex phenomena that have direct analogies with current environmental issues. For example, the behavior sensitive to initial conditions and the inherent unpredictability of these systems can be compared to the effects of climate change, where small changes in environmental factors can lead to significant consequences, such as extreme weather events (Meyer; Lima, 2023). In this context, mathematics acts as a tool for understanding and a means of action and intervention in socio-environmental problems.

The use of mathematical models applied to everyday problems facilitates the connection between mathematical concepts and students' life experiences. Khotimah and Masduki (2016) highlight that "Contextual Learning" allows students to relate mathematical problems to real situations, promoting a deeper and more critical understanding of knowledge. By modeling the nonlinear pendulum, students can investigate how variables such as friction, external forces, and excitation amplitude influence the behavior of the system, drawing parallels with the accumulation of greenhouse gases and their impacts on the global climate. Thus, as Skovsmose (2007) points out, mathematics in action allows students to perceive the influence of mathematics on decisions that affect society and the environment.

The integration of digital technologies and computer simulations enriches the teaching of differential equations and mathematical modeling, providing interactive learning environments that promote active student participation. Cevikbas, Greefrath, and Siller (2023) argue that the use of digital technologies facilitates experimentation and model refinement, deepening understanding through direct exploration and manipulation of the systems studied. Tools such as GeoGebra allow the exploration of multiple representations and balance analytical, graphical, and numerical approaches, facilitating the understanding of nonlinear phenomena (Igliori; Almeida, 2017). In this sense, mathematics in action is materialized through the use of technologies that allow the simulation and visualization of complex phenomena, making learning more meaningful.

The "Problem-Based Learning" approach places students as active agents in their learning process, using real problems as a starting point (Santos et al., 2020). When facing challenges related to the modeling of the nonlinear pendulum and its socio-environmental implications, students develop autonomy and reflective thinking, aligning with the principles of critical mathematics education defended by Freire (1996) and with the idea of Skovsmose (2007) that mathematics should enable students to actively participate in society.

Mathematical modeling in socio-environmental contexts also promotes the development of students' critical and reflective thinking. Armutcu and Bal (2023) highlight that utilizing mathematical modeling activities in the STEM context significantly increases students' skills, encouraging complex problem-solving and collaboration.

By applying these concepts to issues such as sustainable energy resource planning, students are encouraged to consider the unpredictable behavior of systems and to reflect on viable solutions to global problems. In this way, mathematics in action manifests itself in the students' ability to use mathematical knowledge to influence and understand decisions that have social and environmental impact.

However, it is essential that the integration of digital technologies is accompanied by pedagogical strategies that avoid excessive dependence on automatic solutions, promoting the understanding of the underlying mathematical principles. Andresen (2023) warns of the threat of the "black box", where students blindly trust the solutions generated by the tools, which can limit effective learning. Approaches that encourage independent investigation and exploration of problems can mitigate this risk (Greubel et al., 2022). In this context,

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Skovsmose (2007) emphasizes the importance of developing a critical stance in relation to the use of mathematics, recognizing its active role in modeling and transforming reality.

Teacher education plays a crucial role in the effective implementation of these approaches. Vitoria et al. (2021) emphasize that training strategies that consider the specificities of students and their difficulties with the topic are essential to promote critical and transformative mathematics education. Balseca et al. (2023) add that teacher training is essential to successfully integrate mathematical modeling and digital technologies into the curriculum. In line with this, Skovsmose (2007) argues that educators should be aware of the power of mathematics in action and be able to guide students in critically understanding how mathematics influences and is influenced by social and technological contexts.

We understand that the mathematical modeling of the nonlinear pendulum, using differential equations and computer simulation, presents itself as an effective approach to promote a critical reflection on socio-environmental issues in the teaching of differential equations. By connecting mathematical concepts to real and current contexts, students develop mathematical skills, and also a critical awareness of the environmental challenges that permeate contemporary society. As Skovsmose (2007) emphasizes, mathematics in action is a way to enable students to understand and intervene in the world, recognizing the role of mathematics in the construction and transformation of social and environmental reality.

FINAL CONSIDERATIONS

This study showed that the mathematical modeling of the nonlinear pendulum, using differential equations and computer simulations, can be an effective approach to the teaching of differential equations by promoting a critical reflection on socio-environmental issues. The integration between theory and practice allows students to understand advanced mathematical concepts from their relevance in real contexts, aligning with the principles of Critical Mathematics Education.

By analyzing the different dynamic behaviors generated by variations in the amplitude of external excitation – from regular oscillations to chaotic patterns – the students were able to experience in practice the central characteristics of nonlinear systems. Computer simulation played a crucial role in this process, providing concrete visualizations that facilitated the understanding of complex phenomena and the identification of emerging patterns.

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From the perspective of Freire's (1996) and Skovsmose's (2007) theories, especially the concept of "mathematics in action", the activity allows students to perceive mathematics not only as a set of abstract techniques, but as an active tool that influences and is influenced by social and environmental contexts. By drawing parallels between nonlinear pendulum behavior and phenomena such as climate change, students are encouraged to critically reflect on the impact of small variations on complex systems and the importance of sustainable actions.

The experience revealed that the combination of computer simulation and Critical Mathematics Education creates a dynamic and engaging learning environment. Students have become active agents in their educational process, developing autonomy, reflective thinking, and the ability to connect mathematical concepts to real issues. This is in line with Skovsmose's (2007) idea that mathematics should enable individuals to critically participate in society, understanding and intervening in the processes that shape reality.

However, the study also pointed to the need for caution in the integration of digital technologies. As warned by Andresen (2023), it is critical to avoid over-reliance on automated solutions and ensure that students understand the underlying mathematical principles. Pedagogical strategies that promote independent research and exploration have been shown to be essential to develop a critical stance towards the use of mathematics and technology.

It is concluded that the mathematical modeling of the nonlinear pendulum, supported by computer simulations and based on Critical Mathematics Education, not only enriches the teaching of differential equations but also promotes the formation of critical and conscious citizens. By connecting mathematics to relevant socio-environmental contexts, students are encouraged to use mathematical knowledge as a tool to understand and transform reality, contributing to sustainable development and social justice.

For future initiatives, it is recommended to expand this approach to other nonlinear dynamic systems and to include interdisciplinary projects that address emerging socio-environmental issues. In addition, the continuous training of teachers in Critical Mathematics Education practices and in the effective use of digital technologies is essential for the successful implementation of this methodology, as emphasized by Vitoria et al. (2021) and Balseca et al. (2023).

This work reinforces the importance of a mathematics education that transcends technical learning, promoting critical reflection and the active participation of students in

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society. Mathematics in action, as conceived by Skovsmose (2007) proves to be a powerful approach to enable students to deeply understand contemporary challenges and to act consciously in the construction of a fairer and more sustainable world.

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