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SPATIAL PYTHAGOREAN THEOREM IN TRI-RECTANGULAR TRIHEDRON VIA GRAM MATRIX

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ABSTRACT
The demonstration of this work uses vector calculus and algebra topics to prove the generalized Pythagorean theorem in space using a tri-rectangular trihedron that gives rise to a tetrahedron. The purpose of this paper is to demonstrate that the sum of the squares of the areas of the faces of a tetrahedron is equal to the square of the opposite face. Some concepts of scalar product, Lagrange identity and Gram matrix were widely used. The results showed that its various applications lead to a number of important results and conclusions in Mathematics, Engineering, Science and Geometry. With this, it is concluded that the generalization of the Pythagorean theorem by tri-rectangle trihedron using the Gram matrix and this Important Lagrange Identity, is an important contribution, as it expands its relevance and importance in mathematics and other areas of knowledge.

Keywords: Lagrange's identity, Gram's matrix, Generalization of the Pythagorean theorem.

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INTRODUCTION

The Pythagorean theorem, formulated in the sixth century B.C., represented a fundamental milestone in the history of mathematics and geometry (BOYER, 2012). However, over the centuries, mathematicians from different cultures have sought to generalize this theorem beyond the initial context of right triangles in the plane.

The search for generalizations led to remarkable discoveries, expanding the reach of the theorem to different dimensions and geometric structures. The historical trajectory of this generalization reflects humanity's constant desire to understand and apply fundamental mathematical principles in increasingly abstract and challenging contexts.

In this research work it is intended to demonstrate the Generalized Pythagorean theorem, specifically by means of a tri-rectangle trihedron, where the three-dimensional geometric relations that involve the sides of a solid are explored. Unlike traditional proofs in the plane, this approach expands the application of the theorem to more complex situations, unlike conventional ones.

By examining tri-rectangles trihedrons, which are made up of three orthogonal vectors, one can explore the connections between the lengths of these vectors and investigate how the Pythagorean theorem manifests itself in the three-dimensional context. This contemporary perspective not only enriches our understanding of spatial geometry, but also highlights the versatility and continued applicability of the well-known Pythagorean theorem in broader contexts.

In this way, this research intends to fill a gap in relation to proofs of this generalization of the Pythagorean theorem, since a state of the art was carried out on the theme "Generalization of the Pythagorean Theorem using Tri-Rectangle Trihedron" and no proofs were found using approaches from this research.

This theme referring to new demonstrations on Generalization of the Pythagorean theorem using the Tri-Rectangle Trihedron, arose from the need to deepen the understanding of contemporary Spatial Geometry. As mathematics evolves, new approaches and contexts emerge in which the Pythagorean theorem can be applied, (ABREU, I., 2023)

The investigation of these innovative approaches in tri-rectangle trihedra, as mentioned above, not only enriches theoretical knowledge, but also offers practical insights in fields of physics, engineering, and computer science, where three-dimensional applications are frequent.

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In addition, understanding these generalizations contributes to the expansion of the mathematical repertoire, promoting the discovery of geometric relationships in more complex environments. This research not only preserves the relevance of the Pythagorean theorem, but also highlights its adaptable and vital role in solving contemporary problems.

METHODOLOGY

In this research, a comprehensive review of the mathematical literature was carried out to understand the existing proofs and the approaches already used "State of the Art", on the subject involving the proofs suggested in this work, in order to contribute to a new proof of this theorem in space. In addition, we sought to identify recent studies that have explored the theme, highlighting knowledge gaps or new perspectives.

In a clear and concise way, the definition of the tri-rectangle trihedron was established, delimiting its properties and fundamental characteristics, where it was sought to explore the relationship between the orthogonal vectors that make up the trihedron and its relevance to the Pythagorean theorem in space.

A comparison of the new demonstrations with the traditional approaches, highlighting similarities, differences and proportional advantages by the use of the tri-rectangle trihedron were carried out.

The tri-rectangle trihedron is a set of three line segments that intersect at a point and form right angles to each other. One of its most important properties is that the measurements of the edges that make up each plane are in the ratio of 3:4:5, which implies that the trihedron is tri-rectangle. This unique ratio makes it possible to calculate the measurements of the unknown edges of a tri-rectangle trihedron, as long as the measurement of at least one of the edges is known.

The authors (IEZZI, Gelson, 2010) and present a detailed explanation of the tri-rectangular trihedron and its properties. One of the statements of this reference is that the ratio 3:4:5 is unique to a trirectangular trihedron, and that if the edge measurements are not in this ratio, the polyhedron is not a trirectangular trihedron. Also, as mentioned earlier, this ratio can be used to calculate the measurements of the unknown edges of a trirectangular trihedron, as long as the measurement of at least one of the edges is known. This makes the tri-rectangular trihedron even more useful for calculations in areas involving spatial geometry.

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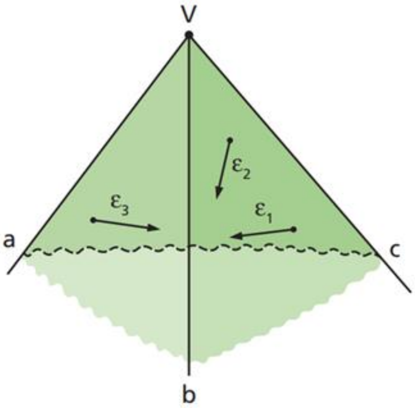
It is important to emphasize that understanding the properties of the tri-rectangular trihedron is fundamental for a good performance in spatial geometry. Some of the concepts and elements related to trihedra will be presented below according to (DOLCE, 2013b Elon, 2014).

A trihedron is defined as the intersection of three non-coplanar semispaces that have the same origin. These semispaces are represented by , and , which contain the semilines , and respectively. The trihedron is represented by and is given by the intersection of the three semispaces (Figure 1): $V\varepsilon_1\varepsilon_2\varepsilon_3V_aV_bV_cV(a, b, c)$

Figure 1-Trihedron: Dolce and Pompeu (2013).
Intersection of semispaces



It is possible to define the trihedron as the union of three angular sectors formed by the semi-straight lines , and . So it is called the trihedral sector or three-edged solid angle. The orientation of the trihedron may vary, but it will always be the intersection of the three semispaces or the union of the three angular sectors (Camargo; Carvalho, 2007). Elements of a trihedron (Figure 2):$V_aV_bV_c$

a) The elements of a trihedron include the vertex , the three edges , , and the three faces or face angles: e (or and).$VV_aV_bV_ca\widehat{V}b, a\widehat{V}cb\widehat{V}c\widehat{ab}, \widehat{ac}\ \widehat{bc}$
b) The dihedral of the trihedron are denoted by and each of them is determined by two faces of the trihedron.$di(a), di(b), di(c)$
c) A trihedron section is a triangle with a single vertex on each edge.ABC

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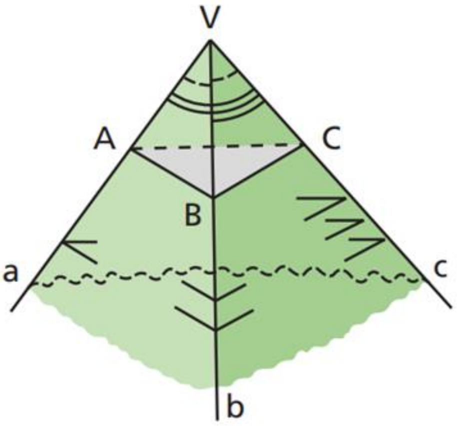
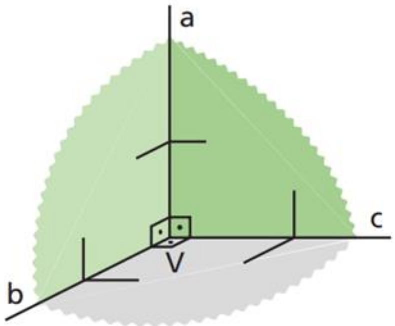
Figure 2-Dolce and Pompeu (2013b).
Elements of a trihedron



A notable trihedron is the tri-rectangle trihedron (or tri-rectangular trihedron), which has right-angled faces and right dihedral (Figure 3).

Figure 3 - Dolce and Pompeu (2013a).
Remarkable trihedron



The presence of the tri-rectangular trihedron in spatial geometry is widely studied and explored in different books, demonstrating the relevance of this polyhedron in mathematics and practical areas. The study of the tri-rectangular trihedron allows the solution of problems involving calculations of unknown measurements in polyhedra, making it a useful tool for applications in areas such as engineering, architecture and art, (DOLCE, Osvaldo, 2013b).

As previously stated, the Pythagorean theorem is one of the most important and fundamental discoveries in mathematics, and has been used by countless generations of students and professionals around the world. The theorem, as we have seen, states that, in a right triangle, the square of the hypotenuse (the side opposite the right angle) is equal to the sum of the squares of the legs (the other two sides).

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The tri-rectangular trihedron, in turn, is a set of three planes that intersect at a right angle and form a three-dimensional Cartesian coordinate system. This system is fundamental to analytic geometry and to several areas of mathematics and physics.

The relationship between the Pythagorean theorem and the tri-rectangular trihedron is quite interesting and relevant, since the theorem can be used to calculate the distance between two points in a three-dimensional coordinate system, (CAMARGO, 2007). To understand this relationship, we can imagine a point on the tri-rectangular trihedron with coordinates and another point Q with coordinates $\mathrm{P}(\mathrm{x},\mathrm{y},\mathrm{z}),(\mathrm{x}',\mathrm{y}',\mathrm{z}')$.

The distance between these two points is given by:

$$\mathrm{d}=\sqrt{(\mathrm{x}'-\mathrm{x})^2+(\mathrm{y}'-\mathrm{y})^2+(\mathrm{z}'-\mathrm{z})^2}$$

This formula can be easily derived from the Pythagorean theorem (STEINBRUCH, Winterle, 1991). In addition, the tri-rectangular trihedron can also be used to visualize the relationships between the different angles and sides of a right triangle. For example, one can imagine a right triangle with the hypotenuse and legs and . $\mathrm{ABCABACBC}$

If you draw a plane that contains the hypotenuse and the line perpendicular to it that passes through the point, you have one of the planes of the trirectangular trihedron.ABC

This plane will also contain the right angle, and you can use the trigonometric relations of the right triangle to calculate the values of the other angles and sides. For example, the tangent of the angle to calculate the measure of the opposite leg, or the cosecant of the angle to calculate the measure of the hypotenuse.$\mathrm{CC\ ACCAB}$

Thus, the relationship between the Pythagorean theorem and the tri-rectangular trihedron is fundamental for analytic geometry and for visualizing the relationships between the different angles and sides of a right triangle in three dimensions. This relationship is widely explored in various fields of mathematics and physics, and can be found in several bibliographic references, such as (LAGES, Elon, 2006).

The following is the mathematical basis that was used to develop the research and the demonstration of the Generalized Pythagorean Theorem that are applicable to the tri-rectangular trihedron.

Given two vectors $u,v\in\mathbb{R}^3$ in vector notation or in the form of coordinates: and : , is called the scalar product of these vectors, and is represented by , to the real number$\vec{u}=x_1\vec{i}+y_1\vec{j}+z_1\vec{k}=(x_1,y_1,z_1)\vec{v}=x_2\vec{i}+y_2\vec{j}+z_2\vec{k}=(x_2,y_2,z_2)\vec{u}.\vec{v}$

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$$\vec{u}.\vec{v} = < \vec{u}, v \geq = x_1x_2 + y_1y_1 + z_1z_2. \quad (1)$$

It is called the two-vector vector product , and , taken in that order, and is represented by x to the vector$\vec{u}, \vec{v} \in \mathbb{R}^3 \vec{u} = x_1\vec{\imath} + y_1\vec{\jmath} + z_1\vec{k} = (x_1, y_1, z_1)\vec{v} = x_2\vec{\imath} + y_2\vec{\jmath} + z_2\vec{k} = (x_2, y_2, z_2)$ $\vec{u}\vec{v}$

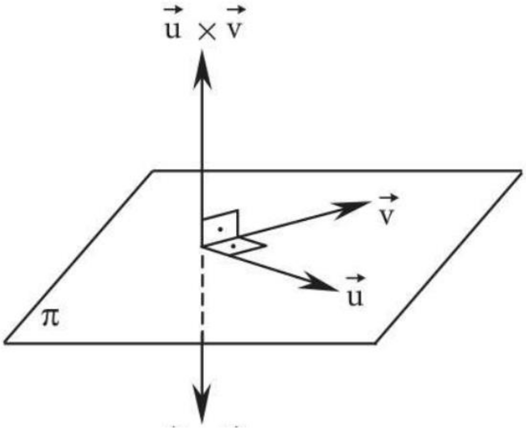
$\vec{u}$ x .

$$(2)\vec{v} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}\vec{\imath} - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}\vec{\jmath} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}\vec{k}$$

Note that the vector product of is also indicated by and reads " vector " and that the definition of x , given in (2) can be obtained in development according to Laplace's theorem. Therefore, x is a vector, $\vec{u}$ por $\vec{v}\vec{u}\wedge\vec{v}\vec{u}$ $\vec{v}\vec{u}\vec{v}\vec{u}\vec{v}$ (WINTERLE, Paulo, 2014).

Important characteristics of the vector product are mentioned below x a) $\vec{u}\vec{v}$:With respect to its direction, this vector x is simultaneously orthogonal to the vectors and . See Figure (3).$\vec{u}\vec{v}\vec{u}\vec{v}$

Figure 4: $\vec{u}$ x $\vec{v}$ orthogonal to vectors and $\vec{u}\vec{v}$.
Vector product between two vectors



b)) The length of the vector $\vec{u}$ x $\vec{v}$ is given by the following expression, where the angle between the vectors and non-zeros is:$\theta\vec{u}\vec{v}$

$$|\vec{u} \times \vec{v}\ | = |\vec{u}\ || \vec{v}\ |sin\theta \quad (3)$$

c) Lagrange identity

$$|\vec{u}\ x\ \vec{v}|^2 = |\vec{u}|^2.|\vec{v}|^2 - (\vec{u}.\vec{v})^2 \quad (4)$$

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DEMONSTRATION

To prove this identity, we use the results already known from the remarkable products, which is the square of the difference between two terms:

$$(a-b)^2 = a^2 + b^2 - 2ab. \quad (5)$$
$$(a-b)^2 + (c-d)^2 = a^2 + b^2 + c^2 + d^2 - 2(ab - cd) \quad (6)$$

From Lagrange's Identity, Equation (4), we will denote the first member of this equality by , i.e., , and the terms and by and , respectively. The idea will be to show that to prove identity.$(I)|\vec{u}\ x\ \vec{v}|^2|\vec{u}|^2.|\vec{v}|^2(\vec{u}.\vec{v})^2(II)(III)(I) = (II) - (III)$

From Equality , that is, , the definition of vector product is developed between vectors in vector notation or in the form of coordinates, according to Equation (2):$(I)|\vec{u}\ x\ \vec{v}|^2\vec{u}$ e $\vec{v}$

$$\vec{u}\ \text{x}\ .\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}\vec{i} - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}\vec{j} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}\vec{k}$$
$$= (y_1z_2 - y_2z_1)\vec{i} - (x_1z_2 - x_2z_1)\vec{j} + (x_1y_2 - x_2y_1)\vec{k}$$
$$= \big((y_1z_2 - y_2z_1), (x_1z_2 - x_2z_1), (x_1y_2 - x_2y_1)\big). \quad (7)$$

Taking the module of the vector product among the vectors, we get:$\vec{u}$ e $\vec{v}$

$$|\vec{u}\ \text{x}\ \vec{v}| = \sqrt{(y_1z_2 - y_2z_1)^2 + (x_1z_2 - x_2z_1)^2 + (x_1y_2 - x_2y_1)^2}$$
$$|\vec{u}\ \text{x}\ \vec{v}|^2 = (y_1z_2 - y_2z_1)^2 + (x_1z_2 - x_2z_1)^2 + (x_1y_2 - x_2y_1)^2$$

Using the result of Equation (5), we have:$(a-b)^2 = a^2 + b^2 - 2ab,$

$$(y_1z_2 - y_2z_1)^2 = (y_1z_2)^2 + (y_2z_1)^2 - 2(y_1z_{2)}(y_2z_1).$$
$$= {y_1}^2{z_2}^2 + {y_2}^2{z_1}^2 - 2y_1z_2y_2z_1. \quad (8)$$
$$(x_1z_2 - x_2z_1)^2 = (x_1z_2)^2 + (x_2z_1)^2 - 2(x_1z_2)(x_2z_{1)}.$$
$$= {x_1}^2{z_2}^2 + {x_2}^2{z_1}^2 - 2x_1z_2x_2z_1. \quad (9)$$
$$(x_1y_2 - x_2y_1)^2 = (x_1y_2)^2 + (x_2y_1)^2 - 2(x_1y_2)(x_2y_{1)}.$$
$$= {x_1}^2{y_2}^2 + {x_2}^2{y_1}^2 - 2x_1y_2x_2y_1. \quad (10)$$

So, it can be stated that (11) $|\vec{u}\ \text{x}\ \vec{v}|^2 = (8) + (9) + (10).$

$$|\vec{u}\ \text{x}\ \vec{v}|^2 = {y_1}^2{z_2}^2 + {y_2}^2{z_1}^2 + {x_1}^2{z_2}^2 + {x_2}^2{z_1}^2 + {x_1}^2{y_2}^2 + {x_2}^2{y_1}^2 - 2(y_1z_2y_2z_1 + x_1z_2x_2z_1 + x_1y_2x_2y_1). \quad (12)$$

Therefore, the first member of the Lagrange Identity, Equation (4), was developed. The next step will be to develop the second member of this identity which is composed of the expression:

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$$|\vec{u}|^2.|\vec{v}|^2 - (\vec{u}.\vec{v})^2. \quad (13)$$

Now we will perform the individual development of each member of Equation (13) and show that Equation (12) is equivalent to Equation (13).

$$|\vec{u}|^2.|\vec{v}|^2 = (x_1{}^2 + y_1{}^2 + z_1{}^2).\,.(x_2{}^2 + y_2{}^2 + z_2{}^2)$$
$$= x_1{}^2x_2{}^2 + x_1{}^2y_2{}^2 + x_1{}^2z_2{}^2 + y_1{}^2x_2{}^2 + y_1{}^2y_2{}^2 + y_1{}^2z_2{}^2 + z_1{}^2x_2{}^2 + z_1{}^2y_2{}^2 + z_1{}^2z_2{}^2. \quad (14)$$

Using the usual scalar product definition, we have:

$$\vec{u}.\vec{v} = \vec{u} = (x_1\vec{\imath} + y_1\vec{\jmath} + z_1\vec{k}).\vec{v}(x_2\vec{\imath} + y_2\vec{\jmath} + z_2\vec{k}) = (x_1, y_1, z_1).(x_2, y_2, z_2) \triangleq x_1x_2 + y_1y_2 + z_1z_2.$$
$$(\vec{u}.\vec{v})^2 = (x_1x_2 + y_1y_2 + z_1z_2)^2 = (x_1x_2 + y_1y_2 + z_1z_2)(x_1x_2 + y_1y_2 + z_1z_2). \quad (15)$$

Applying the Distributive Property to Equation (15) and rearranging, comes

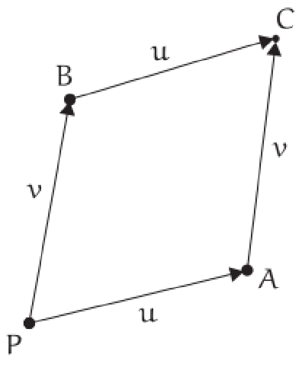
$$(\vec{u}.\vec{v})^2 = x_1{}^2x_2{}^2 + y_1{}^2y_2{}^2 + z_1{}^2z_2{}^2 + 2(x_1x_2y_1y_2 + y_1y_2z_1z_2 + z_1z_2x_1x_2). \quad (16)$$
$$|\vec{u}|^2.|\vec{v}|^2 - (\vec{u}.\vec{v})^2 = (\text{Equação}(14) - \text{Equação}(16)):$$
$$|\vec{u}|^2.|\vec{v}|^2 = x_1{}^2y_2{}^2 + x_1{}^2z_2{}^2 + y_1{}^2x_2{}^2 + y_1{}^2z_2{}^2 + z_1{}^2x_2{}^2 + z_1{}^2y_2{}^2 - -2(x_1x_2y_1y_2 + y_1y_2z_1z_2 + z_1z_2x_1x_2). \quad (17)$$

Comparing Equations (12) and (17), it can be deduced that Equation (4), Lagrange's Identity, is demonstrated.

To use the definition of the Gram matrix in this demonstration, we will assume that for the area of a parallelogram, three (3) non-collinear points in space are considered: .
$P, A, B \in \mathbb{R}^3$

By doing and and considering, we obtain the parallelogram , in which , and . See Figure (4) below:$\vec{u} = \overrightarrow{PA}\vec{v} = \overrightarrow{PB}\ \ \vec{u} + \vec{v} = \overrightarrow{PC}PACBA = P + \vec{u}B = P + \vec{v}C = P + (\vec{u} + \vec{v})$

Figure 4: Parallelogram determined by the point P and the vectors. $\vec{u}, \vec{v}$(WINTERLE, Paulo, 2014)
Parallelogram $PACB$



The Gram matrix of vectors is by definition the expression:$\vec{u}$ e $\vec{v}$

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$$g(\vec{u},\vec{v}) = \begin{bmatrix} \langle \vec{u},\vec{u}\rangle & \langle \vec{u},\vec{v}\rangle \\ \langle \vec{v},\vec{u}\rangle & \langle \vec{v},\vec{v}\rangle \end{bmatrix}. \quad (18)$$

Without any difficulty, it is shown that , see $\langle \vec{u},\vec{v}\rangle = \langle \vec{v},\vec{u}\rangle$(STEINBRUCH, Alfredo, 1991).

In the reference (LAGES, Elon, 2014) it is shown that the determinant of the Gram matrix, , is the square of the parallelogram area See Figure (4). The Analytical Expression of the Theorem follows:$det.\,g(\vec{u},\ \vec{v})\ \ PACB$.

$$det.\,g(\vec{u},\ \vec{v})\ = \left(Área\ do\ paralelogramo\ PACB\right)^2.$$

In view of the exposition of the above contents, we will consider a tri-rectangle trihedron with vertex *O*, cut by any plane, forming the OABC tetrahedron as shown in Figure (5). Be , and . The objective is to demonstrate that the square of the area of triangle $S_1 = Área\ do\ \triangle_{ABC}\ S_2 = Área\ do\ \triangle_{OBC}\ S_3 = Área\ do\ \triangle_{OCA}$*ABC* is equal to the sum of the squares of the areas of the other three triangles, using the Lagrange Identity in the matrix and Gram.

In other words, we can say that the aim of this work is to show analytically that the application of Lagrange's identity together with the Gram matrix, leads to the Generalized Pythagorean Theorem using the tri-rectangle trihedron. In reference (LAGES, Elon, 2006)., it is stated that "The square of the area of triangle *ABC* is equal to the sum of the squares of the areas of the other three triangles". In previous research paper, it was shown that this same problem was demonstrated using (ABREU, I., 2023).

Proof (Generalized Pythagorean Theorem in the Trihedron-Rectangular using the Gram Matrix).

Consider the vector product of the vectors " and " given by$\vec{u}\ \vec{v}$

$$\vec{u}\ \text{x . So}\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}\vec{i} - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}\vec{j} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}\vec{k}$$

$$|\vec{u}\ \text{x}\ \vec{v}|^2 = (y_1z_2 - y_2z_1)^2 + (x_1z_2 - x_2z_1)^2 + (x_1y_2 - x_2y_1)^2. \quad (19)$$

On the other hand, vector Gram matrix is by definition the expression:$\vec{u}$ e $\vec{v}$

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$$g(\vec{u},\vec{v}) = \begin{bmatrix} \langle\vec{u},\vec{u}\rangle & \langle\vec{u},\vec{v}\rangle \\ \langle\vec{v},\vec{u}\rangle & \langle\vec{v},\vec{v}\rangle \end{bmatrix} = \begin{bmatrix} {x_1}^2 + {y_1}^2 + {z_1}^2 & x_1x_2 + y_1y_2 + z_1z_2 \\ x_2x_1 + y_2y_1 + z_2z_1 & {x_2}^2 + {y_2}^2 + {z_2}^2 \end{bmatrix} \quad (20)$$

Without any difficulty, it is shown that, usual internal product, see Equation (1).$\langle\vec{u},\vec{v}\rangle = \langle\langle\vec{v},\vec{u}\rangle$

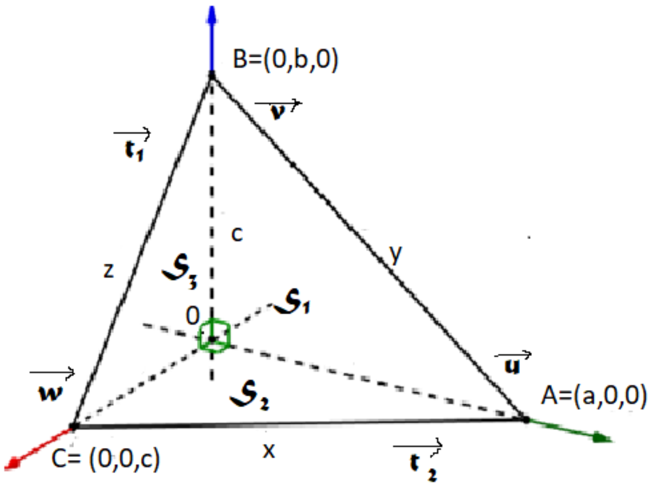
It follows that:

$$det.\, g(\vec{u},\vec{v}) = ({x_1}^2 + {y_1}^2 + {z_1}^2).({x_2}^2 + {y_2}^2 + {z_2}^2) - (x_1x_2 + y_1y_2 + z_1z_2)^2$$
$$= |\vec{u}|^2.|\vec{v}|^2 - (\vec{u}.\vec{v})^2 = |\vec{u}\ x\ \vec{v}\ |^2 \text{ (Lagrange's identity)}$$

Like this. (21)$det.\, g(\vec{u},\ \vec{v})\ = (Área\ do\ paralelogramo\ PACB)^2$

For this new proof of the Generalized Pythagorean Theorem, we use Figure (5) of the Tri-rectangular Trihedron on the space-coordinated axes, *Ox*, *Oy*, and *Oz*, and assume that . For this purpose, consider the following classical relationships of the three right triangles involved in the figure. Let and the areas of the right triangles ABC$A = (0, c, 0), B = (b, 0, 0)\ e\ C = (0, 0, a)S, S_1, S_2S_3$, *AOC*, *BOC* and *AOB* be Still for Figure (2), consider the vectors on the coordinate axes.$\vec{u} = (b, o, o), \vec{v} = (0,0,a)$ e $\vec{w} = (0, c, 0)$

FIGURE 5: TRI-RECTANGULAR TETRAHEDRON ON THE AXES
Tetrahedron on axes



So it can be deduced from Figure 1 that $S = \frac{1}{2}(Área\ do\ paralelogramo\ PACB)$

$$Área\ do\ paralelogramo\ PACB = 2S. \quad (22)$$

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Substituting Equation (22) in Equation (21), we have that:

. (23)$det.\,g(\vec{u},\vec{v}) = (2S)^2 = 4S^2$

Note that S is the area of , and it should be shown that the square of the area of triangle ABC is equal to the sum of the squares of the areas of the other three triangles in Figure 3. From this it follows that Δ_{ABC}

$S^2 = \frac{1}{4} det.$ (24)$g(\overrightarrow{t_1}, \overrightarrow{t_2})$.

Then deduce the following relations for areas of the other three triangles:

$S_1{}^2 = \frac{1}{4} det.\quad .g(\vec{u},\vec{v}) \rightarrow 4S_1{}^2 = det.\,g(\vec{u},\vec{v})$
$S_2{}^2 = \frac{1}{4} det.$ (+) (25)$g(\vec{u},\vec{w}) \rightarrow 4S_2{}^2 = det.\,g(\vec{u},\vec{w})$.
$S_3{}^2 = \frac{1}{4} det.\,g(\vec{v},\vec{w}) \rightarrow \quad 4S_3{}^2 = det.\,g(\vec{v},\vec{w})$.
$4S_1{}^2 + 4S_2{}^2 + 4S_3{}^2 = det.\,g(\vec{u},\vec{v}) + det.\,g(\vec{u},\vec{v}) + det.\,g(\vec{v},\vec{w})$.
$4(S_1{}^2 + S_2{}^2 + S_3{}^2) =$ (26)$det.\,g(\vec{u},\vec{v}) + det.\,g(\vec{u},\vec{v}) + det.\,g(\vec{v},\vec{w})$.

At this point, the Gram matrices formed by the vectors and $\vec{u}, \vec{v}\vec{w}$.

$$det.\,g(\vec{u},\vec{v}) = \begin{vmatrix} \langle\vec{u},\vec{u}\rangle & \langle\vec{u},\vec{v}\rangle \\ \langle\vec{v},\vec{u}\rangle & \langle\vec{v},\vec{v}\rangle \end{vmatrix} = \begin{vmatrix} a^2 & 0 \\ 0 & b^2 \end{vmatrix} = a^2b^2.$$
$$det.\,g(\vec{u},\vec{w}) = \begin{vmatrix} \langle\vec{u},\vec{u}\rangle & \langle\vec{u},\vec{w}\rangle \\ \langle\vec{w},\vec{u}\rangle & \langle\vec{w},\vec{w}\rangle \end{vmatrix} = \begin{vmatrix} a^2 & 0 \\ 0 & c^2 \end{vmatrix} = a^2c^2.$$
$$det.\,g(\vec{v},\vec{w}) = \begin{vmatrix} \langle\vec{v},\vec{v}\rangle & \langle\vec{v},\vec{w}\rangle \\ \langle\vec{w},\vec{v}\rangle & \langle\vec{w},\vec{w}\rangle \end{vmatrix} = \begin{vmatrix} b^2 & 0 \\ 0 & c^2 \end{vmatrix} = b^2c^2.$$

From this it can be deduced that from Equation (26) the following result:

$4(S_1{}^2 + S_2{}^2 + S_3{}^2) = a^2b^2 + a^2c^2 + b^2c^2.$ (27)

Finally, we calculate the frontal area S of what is formed by the vectors

Hence:$\Delta_{ABC}\overrightarrow{t_1} = (0, b, -c)\ e\ \overrightarrow{t_2} = (a, 0, -c)$.

$4S^2 = \frac{1}{4} det.\,g(\overrightarrow{t_1}, \overrightarrow{t_2}) = \frac{1}{4} det \begin{vmatrix} \langle\overrightarrow{t_1},\overrightarrow{t_1}\rangle & \langle\overrightarrow{t_1},\overrightarrow{t_2}\rangle \\ \langle\overrightarrow{t_2},\overrightarrow{t_1}\rangle & \langle\overrightarrow{t_2},\overrightarrow{t_2}\rangle \end{vmatrix}$
$4S^2 = det \begin{vmatrix} b^2 + c^2 & c^2 \\ c^2 & a^2 + c^2 \end{vmatrix} = (b^2 + c^2).(a^2 + c^2) - c^4$
$= b^2a^2 + b^2c^2 + c^2a^2 + c^4 - c^4.$
$4S^2 = a^2b^2 + a^2c^2 + b^2c^2.$ (28)

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Comparing Equations (27) and (28), we have:

$$4S^2 = 4\left({S_1}^2 + {S_2}^2 + {S_3}^2\right) \rightarrow S^2 = {S_1}^2 + {S_2}^2 + {S_3}^2 \quad \text{cqd.} \tag{29}$$

From this we can conclude with this last result presented by Equation (29) that we have achieved our goal.

RESULTS

In this section, the main findings in the development of the work are presented and interpreted, relating them to the proposed objectives. In the case of the topic Generalized Pythagorean Theorem in the Tri-Rectangular Trihedron via Gram Matrix, the results emphasize both the theoretical aspects and their possible applications. Here are the topics that could be developed: Mathematical formulation of the Generalized Theorem, and the general discovery developed for the Generalized Pythagorean Theorem in the trirectangular trihedron was demonstrated. The above expression was related to the Gram Matrix and showed how it simplifies the calculations by ensuring that, in the case of orthogonal vectors, the terms outside the diagonal are null. It is explained how the determinant of the Gram Matrix, in this specific case, can be used to verify the orthogonality and calculate the volume of the parallelepiped generated by the vectors. Regarding the Generality of the Method, it was demonstrated that the formulation via Gram Matrix is valid for larger dimensions, with the sum of the squares of the norms of the orthogonal vectors remaining true regardless of the number of dimensions. Cases were presented in which the Gram matrix is used to calculate volumes or areas in more complex situations, such as in spaces of higher dimensions or with non-orthogonal vectors. The simplicity and characteristic of the method to generalize the Pythagorean Theorem, the contribution of the use of the Gram Matrix to unify the algebraic and geometric analysis and the robustness of the method in three-dimensional contexts and its extensions consolidate the methodology in applications in various areas of everyday life.

DISCUSSION

Regarding the discussion, the interpretation with reference to the following points should be highlighted:

- Their interpretation: The explanation of how the Gram Matrix provides a generalization of the Pythagorean Theorem can be structured in three parts: geometric intuition, the role of the Gram Matrix, and generalization in the three-dimensional context. This can be done by using the Pythagorean theorem in the plane. Geometrically, this reflects the sum of the areas of the squares built on the legs being equal to the area of the square built on the hypotenuse. This result is directly related to the orthogonality of the vectors associated with the sides of the triangle. With respect to Generalization to Greater Dimensions, as in a three-dimensional space, the concept of orthogonality extends to vectors that are mutually perpendicular (in the case of a trirectangular trihedron). In this configuration, the Pythagorean theorem can be generalized to include the sum of squares of the norms of three orthogonal vectors.
- Relation to the application of the Gram Matrix to geometry: while the classical Pythagorean Theorem is limited to the relationships of lengths in the plane, the Gram Matrix offers a tool that generalizes the ideas of the theorem to multiple dimensions and to situations where orthogonality and more complex metrics are easy. This connects the theorem with advanced linear algebra concepts.
- The relevance of the use of the Gram Matrix in the Generalized Pythagorean Theorem in the Tri-Rectangular Trihedron is manifested in an abstract and generalized way, since the Gram Matrix offers a systematic and algebraic way of studying geometric properties such as, vector lengths, angles between vectors (through the internal products), volumes of geometric figures. The connection between Linear Algebra and Geometry is clearly highlighted. Among the numerous applications, Computer graphics and 3D Modeling, Engineering and Physics, Data Science and Machine Learning can be highlighted.

The Gram matrix provides a generalization because it abstracts the central idea of the Pythagorean theorem (relations based on orthogonality and sum of squares) to any set of vectors and dimensions, preserving the link between linear algebra and geometry.

The use of the Gram Matrix transcends the simple generalization of the Pythagorean Theorem in three-dimensional space. It reveals itself as a powerful tool, unifying algebra and geometry, enabling practical applications in various areas and opening the way for more sophisticated analysis in mathematics, science and technology. Its relevance lies in its ability to expand classical concepts, such as the Pythagorean theorem, to more general and high-dimensional contexts, while maintaining elegance and algebraic simplicity.

CONCLUSION

The study of the Generalized Pythagorean Theorem in the Tri-Rectangular Trihedron via Gram Matrix demonstrates that the presented approach proved to be especially relevant for its simplicity and efficiency, while offering a robust tool for the analysis of complex geometric properties, such as volumes and angles in larger dimensions. In addition, the Gram Matrix proved to be useful to unify fundamental concepts of algebra and geometry, enabling applications ranging from the calculation of volumes to the verification of orthogonality and finally, this work reinforces that the use of the Gram Matrix not only generalizes the Pythagorean Theorem, but also opens new possibilities of application in several areas of science and technology. This approach combines modern math and practicality, offering deep insights and extending the reach of classical concepts to content.

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